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LEVEL II

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NUMERICAL OPTIMIZATION OF SEARCH FOR A MOVING TARGET,

Report to

Naval Analysis Programs
Office of Naval Research

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Under Contract No. N00014-76-C-0696

by

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ABSTRACT

The problem of computing an optimal search plan for a moving target is addressed when the searcher can distribute his effort as finely as he wishes.

The majority of the report describes numerical techniques which have been developed to compute optimal search plans for the very broad class of problems in which the target's motion can be modeled by a discrete-time stochastic process and the detection function is exponential.

A very efficient algorithm is given to find optimal search plans when the target's motion is modeled by a mixture of discrete time and space Markov processes. A second algorithm is presented to solve the variation of this problem that one encounters when the search effort at each time period is restricted to be uniform over an arbitrary rectangular region. The latter is intended to approximate the problem of choosing a sequence of sonobuoy fields to maximize the probability of detecting a submarine. Examples show that one can often find rectangular plans that are almost as effective as the optimal plan. In addition to the above, an algorithm is presented to find optimal plans for arbitrary discrete time target motion processes which can be modeled by Monte Carlo simulation. All the algorithms have been programmed in FORTRAN and run on a Prime 400 minicomputer. Examples of optimal plans calculated by these algorithms are presented.

All the algorithms are based on a very general necessary and sufficient condition for optimal search for a moving target which is proved in this report. For discrete time and an exponential detection function this condition becomes:

> For a search plan for a moving target to be optimal, it is necessary and sufficient that at each time t it assign an allocation which is optimal for the stationary target problem which one obtains at time t by conditioning on failure to detect after t as well as before t under the plan.

Algorithms are also offered for minimizing mean time to detect, for searches involving non-exponential detection functions, and for survivor search problems.

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PREFACE

This is a report by Daniel H. Wagner, Associates to the Naval Analysis Division of the Office of Naval Research (Code 431) on work performed under ONR Contract No. N00014-76-C-0696. This report summarizes the remarkable progress made since the fall of 1976 in the computation of optimal search plans for moving targets.

We would like to express our appreciation for the excellent cooperation and support that has been given to this work by Mr. J. Randolph Simpson and CDR Ronald James of ONR (Code 431).

In reference [a], S. S. Brown made the key observation that the necessary and sufficient conditions for optimal search for a target moving in discrete space and time with an exponential detection function could be stated in terms of solving a sequence of stationary target problems. In each of these stationary searches the target position distributions are conditioned on non-success of future as well as past search under the plan. This observation allowed him to produce a very efficient algorithm for finding optimal search plans for targets whose motion can be modeled by a mixture of discrete time and space Markov processes. Chapter II of this report is based on reference [b], which is a revised version of Brown's work reported in reference [a]. The work on rectangular search plans discussed in Chapter III is based on work by R. P. Buemi reported in references [c] and [d]. This work makes use of many of the ideas of Brown discussed in Chapter II.

In reference [e], L. D. Stone showed that the condition found by Brown holds when the search space is continuous as well as discrete. This is the basic condition given in Chapter I. In addition he found a generalization of the condition for non-exponential detection functions and continuous time, again involving conditioning upon future as well as past search. This allowed a unified statement of the necessary and sufficient conditions for a very broad class of moving target problems for any combination of discrete or continuous space and discrete or continuous time. This statement is given in Chapter VI.

Using this condition, Stone (reference [f]) devised an algorithm to find optimal search plans when the detection function is exponential and the target motion can be modeled by an arbitrary discrete-time Monte Carlo simulation. This algorithm was implemented by C. R. Hopkins (see reference [g]) on a Prime 400. Hopkins' program was used to produce the examples in Chapter IV.

SUMMARY

This report summarizes the recent progress that has been made in finding optimal se.‾ch plans for moving targets.

In the recent past, our ability to compute optimal search plans for detecting moving targets was limited to special situations such as those involving target motion that is conditionally deterministic with a factorable Jacobian or that is two-celled Markovian (see Chapters 8 and 9 of reference [h]). Although the target motion in some Naval applications can be modeled by the factorable conditionally deterministic motion, most cannot, and Markovian motion in any small number of cells is not a realistic motion model for most operational situations.

Since the fall of 1976 we have developed optimization techniques which allow one to find optimal allocations for multiscenario Markovian motions in discrete space and time. These multiscenario Markovian motions are general enough to model a wide range of target motions. In addition, we have developed an optimization program for arbitrary discrete time target motion processes that can be represented by a Monte Carlo simulation. This latter program can be coupled with computer assisted search programs such as COMPASS, MEDSEARCH, or TARDIST which are now being used on a regular basis to provide search planning advice for actual submarine searches in the Mediterranean and Atlantic. The program designed for arbitrary discrete time target

motions is very general; however, it pays the price of being slower to run than the program designed only for multiscenario Markovian motions. Programs such as COMSUBPAC's ASP and SASP could be modified to incorporate the basic optimization algorithm described in the third section below.

The class of search problems that can be solved by these programs is very broad but there are two important restrictions. First the detection function must be exponential, and second the searcher is assumed to have the ability to spread his effort over large areas in a single time period. Thus, the allocations obtained from these programs would not be appropriate for a submarine searching for a submarine but would be more reasonable for a VP aircraft searching for a submarine. Even in the latter case, the optimal allocations of effort are probably too complicated for operational use, although they can suggest the general nature of the plan to be followed. In Chapter III of this report we address this problem by producing an algorithm that finds near-optimal plans which are restricted to allocate their effort uniformly over a single rectangle during each time period. In fact, the methods described in Chapter III have been adapted by COMPATWINGSPAC for use in VP search planning.

In the remainder of this summary, we describe the basic search problem and outline the results obtained for this problem.

Basic Problem

The target's motion is represented by a stochastic process

$$X = \{X_t; t = 1, \ldots, T\}$$

where X_t is a random variable which gives the target's position at time t. The distribution of X_t is simply the probability distribution of the target's location at time t.

In this report, two basic types of target motion processes are considered. The first type, considered in Chapters II and III, is the multi-scenario Markovian motion in discrete space and time. The second type is an arbitrary discrete time target motion process which is represented by a large but finite number of sample paths from a Monte Carlo simulation of the process. This type is considered primarily in Chapter IV. This second type of motion is very general in character; it need not be Markovian and can take place in a discrete or continuous space.

For most of the report, we assume that a grid has been established on the search space which is two-dimensional and that search effort must be applied uniformly within a given cell of the grid, although effort may vary from cell to cell. Let J denote the set of cells in the grid and

$$p_t(j) = \Pr\{X_t \in \text{ cell } j\} \qquad \text{for } j \in J,\ t = 1, \ldots, T,$$

$$A(j) = \text{area of } j^{th} \text{ cell for } j \in J.$$

Although we impose this grid structure, the target motion may take place in either discrete or continuous space. We assume that the detection function is exponential, i.e.,

$$1 - \exp\left(-\frac{W(j)z}{A(j)}\right)$$

is the probability of detecting the target with z amount of effort placed in the j^{th} cell given the target is in that cell. Here, W(j) is the sweep width or effectiveness parameter for search in cell j.

A search plan is described by a nonnegative function ψ of space and time such that

$\psi(j,t)$ = effort placed in cell j at time t for $j \epsilon J$, $t = 1, \ldots, T$.

The amount of effort available is constrained so that

$$m(t) = \text{the effort availabe at time } t.$$

We define $\Psi(m)$ to be the class of search plans ψ such that

$$\sum_{j \epsilon J} \psi(j,t) = m(t) \qquad \text{for } t = 1, \ldots, T.$$

Let E denote expectation taken over the sample paths of the process X. For any search plan ψ the probability, $P_T[\psi]$, of detecting the target by time T is given by

$$P_T[\psi] = E\left[1 - \exp\left(-\sum_{t=1}^{T} W(X_t)\psi(X_t, t)/A(X_t)\right)\right]. \qquad \text{(S-}$$

The basic search problem considered in this report is to find a plan $\psi^* \epsilon \Psi(m)$ that maximizes the probability of detecting the target by time T over all plans in the class $\Psi(m)$. Mathematically stated, we seek $\psi^* \epsilon \Psi(m)$ such that

$$P_T[\psi^*] = \max\{P_T[\psi]: \psi \epsilon \Psi(m)\}. \qquad \text{(S-2)}$$

Such a plan is called T-optimal within $\Psi(m)$.

Basic Necessary and Sufficient Conditions

The main optimization algorithms discussed in this report are based on the following necessary and sufficient condition for a search plan to be T-optimal within $\Psi(m)$. The form of the condition given here applies to any discrete time target motion process and exponential detection function. For any search plan ψ, let

$$g_\psi(j,t) = \Pr\left\{ \begin{array}{l} \text{target is in cell } j \\ \text{at time } t \end{array} \middle| \begin{array}{l} \text{failure to detect at all times} \\ \text{other than } t \text{ using plan } \psi \end{array} \right\}.$$

Note that $g_\psi(j,t)$ is conditioned on failure to detect both before and after time t.

BASIC CONDITION: In order that $\psi^* \in \Psi(m)$ be T-optimal, it is necessary and sufficient that the allocation $\psi^*(\cdot, t)$ maximize the probability of detecting a stationary target with distribution $g_{\psi^*}(\cdot, t)$ and effort $m(t)$ for $t = 1, \ldots, T$.

Description of the Basic Algorithm

The algorithms for finding T-optimal plans operate in the following manner. Beginning at time $t = 1$, they find an optimal allocation of $m(1)$ effort for the initial target distribution. For times $t = 2, \ldots, T$, they calculate the posterior target location distribution at time t given failure to detect at all previous times and allocate $m(t)$ effort in a manner that would be optimal for that stationary target problem. The plan resulting from the first pass is the myopic or incrementally optimal plan.

Subsequent passes operate in the following manner for $t = 1, \ldots, T$. The algorithms compute $g_\psi(\cdot, t)$ where ψ represents the most recent allocation obtained. They then reallocate the effort at time t to be optimal for the distribution $g_\psi(\cdot, t)$ and change ψ to reflect that reallocation.

By performing enough passes, one can come as close to the optimal search plan as he wishes.

Multi-Scenario Markovian Motion

Chapter II considers the problem of finding T-optimal plans when the target motion is described by a mixture of discrete time and space Markov chains. Each chain represents a possible target motion scenario. Suppose there are N scenarios. Then for $n = 1, \ldots, N$, the user may specify a Markov chain $X^n = \{X_t^n; t = 1, \ldots, T\}$ to model the motion in that scenario. The n^{th} scenario is given weight α_n where $\Sigma_{n=1}^{N} \alpha_n = 1$. If $p_t^n(j)$ is the probability that the target is located in cell j at time t under scenario n, then the overall probability of the target being located in cell j at time t is

$$p_t(j) = \sum_{n=1}^{N} \alpha_n p_t^n(j).$$

The target motion assumptions are translated into transition functions for the Markov chains. These functions may be time dependent. One can specify the target's initial distribution, i.e., his distribution at time 1, and constrain the target to have any desired distributions at any subset of the remaining times $\{2, \ldots, T\}$. This feature may be used to model geographic constraints, i.e., if the target distribution must squeeze down to funnel through a strait, one can model this by using the constraints.

As well as describing the algorithm used to compute the optimal allocation, Chapter II gives examples of optimal plans calculated by the program that implements the algorithm on a Prime 400 mini-computer.

One of the examples considered is a constrained Markovian fan. The target's initial distribution is circular normal with a standard deviation of 6 miles. It is

centered in the middle of cell (0, 0) and truncated by a 15 mile by 15 mile square whose center coincides with the center of the distribution. The orientation of the square matches that of the grid. The target's motion is single scenario Markovian with the target choosing its course uniformly from 150° to 210° and its speed between 6 and 12 kts with a best speed of 9 kts, which is given a weight of 1.5; the speed distribution is truncated triangular as shown in Figure S-1. Independent draws are made from these distributions each hour. At hour 8 the target is constrained to have a truncated normal distribution which is equal to the initial one translated 60 miles south. If the constraint at hour 8 were removed, this type of problem would be similar to that faced by a VP aircraft trying to redetect a submarine on which contact has been recently lost. Example 1 of Chapter II presents such a case.

Table S-1 shows how the target density changes over the first four hours in the absence of search. The target moves southward and diffuses from its original distribution. During the last four hours the target density is a mirror image of the first four hours. It still moves southward but converges back to a truncated normal distribution.

Ninety units of search effort are available each hour. Table S-2 compares the detection probabilities achieved by the myopic plan at the end of each hour to those achieved by plans which are optimal for 2, 4, 6, and 8 hours. Observe that no optimal plan significantly outperforms the myopic plan. The myopic plan remains near optimal for many searches although Example 3 in Chapter II and Example 3 in Chapter IV give situations in which the optimal plan is significantly better than the myopic. Table S-3 shows the optimal plan for 8 hours. Again only the tables for the first four

FIGURE S-1

TRIANGULAR DISTRIBUTION ON TARGET SPEED

Note: A weight of 1.5 means that the probability density at the best speed is 1.5 times that at the minimum and maximum speeds which have equal probability densities.

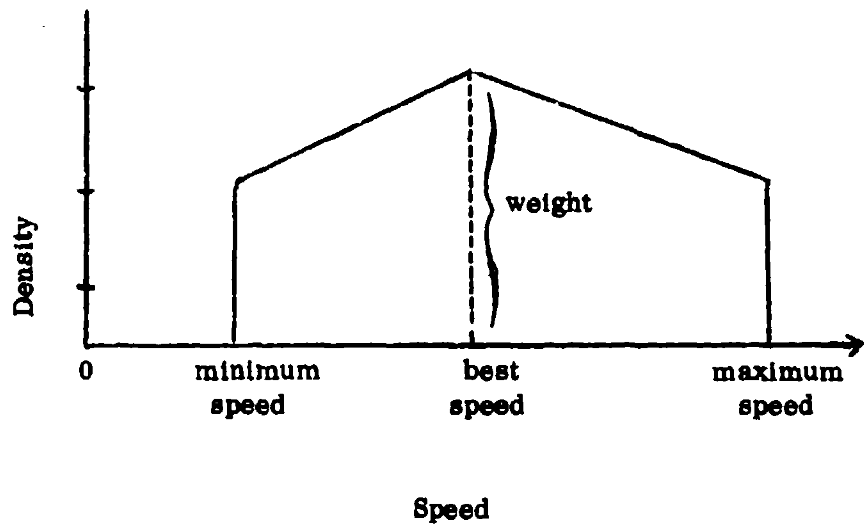


TABLE S-1

TARGET DENSITY FOR THE CONSTRAINED MARKOVIAN FAN

Note: (1) Entries represent thousandths of a target.

(2) Cells are 3 mi x 3 mi.

N

Hour 1

	2	1	0	-1	-2
2]	23	34	38	34	23
1]	34	49	56	49	34
0]	38	56	63	56	38
-1]	34	49	56	49	34
-2]	23	34	38	34	23

Hour 2

	4	3	2	1	0	-1	-2	-3	-4
0]	0	1	3	7	7	7	3	1	0
-1]	0	6	17	31	36	31	17	6	0
-2]	1	9	27	50	58	50	27	9	1
-3]	1	10	30	55	64	55	30	10	1
-4]	1	9	25	47	54	47	25	9	1
-5]	0	4	12	23	26	23	12	4	0
-6]	0	0	2	3	3	3	2	0	0

Hour 3

	4	3	2	1	0	-1	-2	-3	-4
-2]	0	0	1	1	1	1	1	0	0
-3]	0	2	5	9	11	9	5	2	0
-4]	2	7	18	31	37	31	18	7	2
-5]	2	11	29	49	59	49	29	11	2
-6]	3	12	31	53	63	53	31	12	3
-7]	2	9	24	40	48	40	24	9	2
-8]	1	4	10	17	21	17	10	4	1
-9]	0	1	2	3	3	3	2	1	0

TABLE S-1 (continued)

Hour 4

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-5]	0	0	0	1	2	2	2	1	0	0	0
-6]	0	1	3	7	12	14	12	7	3	1	0
-7]	0	2	8	19	33	39	33	19	8	2	0
-8]	0	3	12	30	50	60	50	30	12	3	0
-9]	0	3	12	31	52	61	52	31	12	3	0
-10]	0	2	9	21	36	43	36	21	9	2	0
-11]	0	1	3	8	14	17	14	8	3	1	0
-12]	0	0	1	1	2	3	2	1	1	0	0

Hours 5-8 are a mirror reflection of hours 1-4.

TABLE S-2

DETECTION PROBABILITIES FOR THE CONSTRAINED MARKOVIAN FAN

Number Of Hours	Myopic Plan	Plan Which is Optimal for 2 hours	4 hours	6 hours	8 hours
1	.357	.346	.324	.313	.311
2	.555	.560	.549	.541	.539
3	.672		.679	.673	.672
4	.750		.762	.758	.757
5	.807			.818	.817
6	.852			.862	.862
7	.889				.898
8	.926				.932

TABLE S-3

SEARCH PLANS FOR EXAMPLE 2

Notes: (1) Entries represent thousandths of a unit of search density (effort/mi^2).
(2) Cells are 3 mi x 3 mi.

Optimal Plan for 8 Hours

N

Hour 1

	2	1	0	-1	-2
2]	448	563	568	563	448
1]	423	361	263	361	423
0]	392	268	165	268	392
-1]	403	338	262	338	403
-2]	416	508	502	508	416

Hour 2

	4	3	2	1	0	-1	-2	-3	-4
0]	0	0	0	0	0	0	0	0	0
-1]	0	0	101	485	532	485	101	0	0
-2]	0	0	338	605	616	605	338	0	0
-3]	0	0	363	602	617	602	363	0	0
-4]	0	0	312	583	599	583	312	0	0
-5]	0	0	0	268	323	268	0	0	0
-6]	0	0	0	0	0	0	0	0	0

Hour 3

	4	3	2	1	0	-1	-2	-3	-4
-2]	0	0	0	0	0	0	0	0	0
-3]	0	0	0	0	0	0	0	0	0
-4]	0	0	257	449	484	449	257	0	0
-5]	0	0	424	546	568	546	424	0	0
-6]	0	0	435	553	580	553	435	0	0
-7]	0	0	363	518	544	518	363	0	0
-8]	0	0	0	223	285	223	0	0	0
-9]	0	0	0	0	0	0	0	0	0

TABLE S-3 (Continued)

Hour 4

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-5]	0	0	0	0	0	0	0	0	0	0	0
-6]	0	0	0	0	52	121	52	0	0	0	0
-7]	0	0	0	302	430	452	430	302	0	0	0
-8]	0	0	74	436	503	523	503	436	74	0	0
-9]	0	0	84	441	512	534	512	441	84	0	0
-10]	0	0	0	343	469	492	469	343	0	0	0
-11]	0	0	0	0	173	240	173	0	0	0	0
-12]	0	0	0	0	0	0	0	0	0	0	0

Hours 5-8 are a mirror reflection of hours 1-4.

Myopic Plan

Hour 1

1]	2	1	0	-1	-2
2]	0	256	381	256	0
1]	256	631	756	631	256
0]	381	756	881	756	381
-1]	256	631	756	631	256
-2]	0	256	381	256	0

Hours 2-3 not shown.

Hour 4

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-5]	0	0	0	0	0	0	0	0	0	0	0
-6]	0	0	0	0	0	26	0	0	0	0	0
-7]	0	0	0	242	505	556	505	242	0	0	0
-8]	0	0	0	402	592	614	592	402	0	0	0
-9]	0	0	0	401	590	607	590	401	0	0	0
-10]	0	0	0	295	566	626	566	295	0	0	0
-11]	0	0	0	0	92	205	92	0	0	0	0
-12]	0	0	0	0	0	0	0	0	0	0	0

Hours 5-8 not shown.

hours are displayed since the last four hours are a mirror image of the first four hours. At hour 1 the optimal plan uses most of its effort to surround the target rather than searching the highest probability cells as the myopic plan does (see the end of Table S-3). By hour 4, however, the optimal plan concentrates its effort in the center of the target distribution.

The conclusion that myopic plans are almost as good as optimal plans in many situations has important operational implications. In general, the optimal plan for time T + T' is not a continuation of the optimal plan for time T. Thus if one searches optimally for four hours, one cannot extend this to an optimal search for eight hours. Optimal plans are time-horizon dependent. However, the myopic plan, which simply maximizes the increase of detection probability at each time increment, can always be continued. In addition, the optimal plans typically pay a penalty in probability of detection at the early hours in order to maximize that probability at time T. With a myopic plan no such penalty is incurred and often the resulting probability of detection at the end of any amount of time is close to optimal. Thus when the myopic plan is close to optimal, as it is in many of the examples calculated for this report, the myopic plan is a good one to use for operational purposes.

Even the myopic plans may be difficult to implement operationally because they often call for fine distributions of effort over large areas. Chapter III considers search plans for multi-scenario target motion when at each time period one is restricted to allocating the available effort uniformly over a rectangle. The size, location, and orientation of the rectangle is chosen by the search planner. This restriction is intended to correspond to allocating sonobuoys uniformly in an area.

An algorithm was developed to find good search rectangles, and it has been programmed on a Prime 400 mini-computer. Chapter III presents some examples of rectangle plans found by this algorithm. In particular, plans were computed for the above example. Table S-4 shows the rectangles obtained for hours 1 through 4. Table S-5 compares the probability of detection obtained from the rectangle plan to that obtained by the myopic plan and the optimal plan for eight hours obtained above. Observe that the probability of detection resulting from the rectangular plan is remarkably close to both the optimal plan and the myopic plan.

Arbitrary Discrete Time Target Motion

In Chapter IV we outline a method for finding T-optimal search plans for any discrete time target motion process that can be represented by a Monte Carlo simulation. Observe that this class of target motions is very broad. For example, the target is not restricted to move among a set of cells as is usually the case when one deals with Markov chain models of target motion. The motion process need not be Markovian or even a mixture of Markov processes. The process can be Gaussian, a constrained diffusion, or a random movement through a network. The program developed to perform the optimization for this class of motion processes can be coupled with the Monte Carlo computer assisted search programs such as COMPASS or MEDSEARCH to find optimal allocations of search effort over any time interval of interest for any target motion processes produced by these programs. In fact, the COMPASS programs were used to generate the target motion process for the examples in Chapter IV.

The accuracy of the optimization will be related to the accuracy with which the Monte Carlo simulation represents the target motion process. The algorithm considers

TABLE S-4

PROBABILITY MAPS AND RECTANGULAR BARRIER SEARCH PLAN FOR THE CONSTRAINED MARKOVIAN FAN

Notes: (1) The rectangular barrier for this time interval is the square which surrounds the entire distribution. Entries represent thousandths of a target.

(2) The cells are 3 mi x 3 mi.

PROBABILITY MAP AND RECTANGLE FOR HOUR 1

N

2	23	34	36	34	23
1	34	49	56	45	34
0	35	56	63	56	38
-1	34	49	56	49	34
-2	23	34	38	34	23
-3					

2 1 0 -1 -2 -3

TIME 1 PROBABILITY OF DETECTION = .329

PROBABILITY MAP AND RECTANGLE FOR HOUR 2

0	1	3	7	7	7	3	1	0
0	6	17	31	36	31	17	6	0
1	9	27	50	58	50	27	9	1
1	10	30	55	64	55	30	10	1
1	9	25	47	54	47	25	9	1
0	4	12	23	26	23	12	4	0
0	0	2	3	3	3	2	0	0

0 -1 -2 -3 -4 -5 -6 -7

4 3 2 1 0 -1 -2 -3 -4 -5

TIME 2 PROBABILITY OF NONDETECTION = .539 (Cumulative)

TABLE S-4 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 3

-2	0	0	1	1	2	1	1	0	0
-3	1	3	7	10	11	10	7	3	1
-4	2	9	19	26	29	26	19	9	2
-5	4	15	31	44	45	44	31	15	4
-6	4	16	33	47	48	47	33	16	4
-7	3	12	26	38	40	38	26	12	3
-8	1	5	12	19	21	19	12	5	1
-9	0	1	2	4	4	4	2	1	0
-10									
4	3	2	1	0	-1	-2	-3	-4	-5

TIME 3 PROBABILITY OF NONDETECTION = .660 (Cumulative)

TABLE S-4 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 4

-5	0	0	1	2	3	3	3	2	1	0	0
-6	0	1	5	10	15	17	15	10	5	1	0
-7	1	4	11	23	32	36	32	23	11	4	1
-8	1	5	15	29	38	42	38	29	15	5	1
-9	1	5	15	29	37	41	37	29	15	5	1
-10	1	4	11	23	32	36	32	23	11	4	1
-11	0	2	5	12	18	21	18	12	5	2	0
-12	0	0	1	2	4	5	4	2	1	0	0
-13											

5 4 3 2 1 0 -1 -2 -3 -4 -5 -6

TIME 4 PROBABILITY OF NONDETECTION = .744 (Cumulative)

TABLE S-5

DETECTION PROBABILITIES FOR THE CONSTRAINED MARKOVIAN FAN

	Probability of Detection for		
Time	Rectangular Plan	Myopic Plan	8-Interval Optimal Plan
1	.329	.357	.311
2	.539	.555	.540
3	.660	.672	.672
4	.744	.750	.757
5	.800	.807	.816
6	.846	.852	.862
7	.884	.889	.898
8	.922	.926	.932

only search allocations that, during one time period, are constant within the cells of a grid chosen by the user. Observe that this is only a restriction on the class of allocations which are considered and is not equivalent to assuming that the target motion takes place in discrete space. The algorithm described in Chapter IV is more general than the one in Chapter II for mixtures of Markovian processes. However, it does in effect optimize over a sample of the target motion process whereas the algorithm in Chapter II treats the target motion exactly. In addition computing time for the more general algorithm is longer than that for a mixture of Markov chains. So, in the case where a mixture of discrete time and space Markov chains provides a good representation of the target's motion, the algorithm in Chapter II will be more accurate and faster.

Three examples of optimal allocations obtained by the general discrete time optimizer are given in Chapter IV. One of these examples involves a fleeing target problem similar to the one described in reference [i], page 17. The target's initial distribution is circular normal with standard deviation 20 miles in any direction. The target is assumed to be traveling at 10 kts along a constant course which is chosen from a uniform distribution on $[0^{\circ}, 360^{\circ}]$. Reference [i] was able to compute the target location distribution for this problem as a function of time when no search effort is applied. However, optimal plans for this problem were not found. Tables IV-1 and IV-2 show the myopic and optimal plans for seven hours of search when the searcher begins his search at hour 4. The optimal allocation concentrates its effort much more heavily in the center of distribution at hour 4 than the myopic plan. Comparing the optimal plan to the myopic for the remaining time intervals, one sees that the optimal plan generally concentrates its effort more than the myopic plan. Also it appears that

at hour 5 the optimal plan chooses to concentrate its search more heavily on the eastern side of the distribution and then to make up for that by concentrating its effort on the western side at hours 6 and 7.

Although the myopic plan and optimal plan are qualitatively different, the resulting probabilities of detection as shown in Table IV-3 are strikingly close. This feature held true for several variations on this fleeing target problem; in particular, when the initial distribution was uniform over a 60 mile by 60 mile rectangle and when the target's speed was allowed to be drawn from a uniform distribution over 6-14 kts.

Example 3 of Chapter IV shows a significant improvement in the optimal plan over the myopic one. This example involves multiscenario target motion and regions of varying detection capability. The detection probabilities for the optimal and myopic plans are shown in Table IV-9. The optimal has 19% better probability of detection than the myopic (i.e., .58 versus .48 for four hours of search).

Algorithms for Related Problems

The method of designing the algorithms in Chapters II and IV can be used with some variations on a wide range of optimal search problems. In Chapter V we outline algorithms which can be used to find optimal plans when the detection function is not exponential, to find plans which minimize mean time to complete a search, and to find optimal allocations of effort when there is a constraint on total effort available but not on the rate at which effort may be applied. We also give an algorithm for maximizing the probability of finding a target alive when the target has a stochastic lifetime which varies with location.

These proposed algorithms have not been programmed or tested. They should be thought of as first approaches to solving these problems, and are included simply to illustrate the methods which can be used to attack these problems.

General Necessary and Sufficient Conditions

Chapter VI finds necessary and sufficient conditions for a search plan to maximize the probability of detecting a moving target by time T under constraints on the rate at which search effort may be applied. These conditions apply to a broad class of moving target problems in continuous or discrete time and in a continuous or discrete space. Many previous results concerning necessary and sufficient conditions for moving target problems are special cases of these results. In particular the basic condition stated in the beginning of this summary is a special case of the results in Chapter VI.

NUMERICAL OPTIMIZATION OF SEARCH FOR A MOVING TARGET

CHAPTER I

INTRODUCTION

Since the fall of 1976, remarkable progress has been made in our ability to compute optimal search plans for moving targets. This report summarizes that progress.

Prior to the work reported here there were few situations in which one could find optimal search plans for moving targets. These situations typically involved two-celled Markovian motion or special types of conditionally deterministic target motion; see Chapters 8 and 9 of reference [h]. In this report we discuss techniques which allow us to find optimal search plans whenever the target motion is modeled by a discrete-time stochastic process in either continuous or discrete space and a fixed amount of effort must be applied at each time period. It is assumed that at each time period the search effort may be distributed as finely as desired over the search space. The optimization techniques presented in this report are primarily limited to exponential detection functions.

Observe that the class of allowable target motions is essentially unrestricted except that a discrete-time motion model must be used. In addition to covering Markovian motion, the class encompasses any of the Monte Carlo target motion processes used by the Navy's computer-assisted search systems such as COMPASS,

MEDSEARCH, or TARDIST, see references [j] and [k].

The assumption that the searcher can distribute his effort as finely as he wishes is an important restriction. It effectively prevents us from using these results to plan the search of one submarine for another. However, the results do apply to searches where the searcher can move much faster than the target, e.g., patrol aircraft searching for a submarine with sonobuoys or aircraft searching visually for a person adrift in a life raft.

The problem of optimal search for a submarine by patrol aircraft motivated much of the work presented in this report. Using the algorithms presented here, one can provide good advice on search allocation for a wide range of problems involving VP aircraft searching for submarines with sonobuoys. In fact the techniques in Chapter III have already been adapted by COMPATWINGSPAC for use in search planning.

Our approach to solving the optimal search problem considered here is to find a set of necessary and sufficient conditions for an optimal plan. We then design an algorithm to find plans which satisfy these conditions and are therefore optimal. For three variations on the basic search problem described in the first section, we have developed and implemented algorithms for a Prime 400 minicomputer. In Chapters II, III, and IV we illustrate the type and complexity of problems that these algorithms can solve.

In the first section of this chapter we describe the basic class of moving target problems that we will consider. In the second section we present the necessary and sufficient conditions which form the basis of the algorithms given in Chapters II and IV. These conditions have a very interesting and useful interpretation in terms of optimal search for stationary targets which is discussed in the second section. The third section describes the algorithm used to compute optimal search plans.

Basic Search Problem

In this section we give a description of the basic search problem which is considered throughout most of this report.

Target motion. The target's motion is represented by a stochastic process $X = \{X_t;\ t = 1, \ldots, T\}$ where X_t is a random variable which gives the target's position* at time t. The marginal distribution of X_t is simply the probability distribution for the target's location at time t.

In a typical search problem, one is given the target's initial probability distribution and a stochastic description of the target's motion. These two can be combined in the manner discussed in Chapters II and IV to produce the stochastic process which represents all the possible target paths over time $t = 1, \ldots, T$.

As an example one might have a bivariate normal distribution for the target's location at time $t = 1$. Such a distribution could be obtained from a long range sensor with poor localization capabilities. From geographical considerations one might be able to deduce that the target's course lies between bearings θ_1 and θ_2 and from operational considerations that the submarine is traveling with a speed between v_1 and v_2. Assume that the probability distributions on target's course and speed are uniform over the above limits. In addition one could assume that the target, having chosen a heading and speed from the above distribution, persists in that course for a random time which is exponentially distributed with specified mean τ_1. At the end of this time, the target makes a new and independent draw from the above distributions to determine its new course and speed.

---

* We understand X_t to be either a point in the plan or a cell index. It will be clear from context which is the case.

Using the COMPASS or MEDSEARCH programs, this type of target motion is represented by producing a large number of target paths (see reference [j]) in a Monte Carlo fashion. From these paths, the programs can generate a sequence of distributions for the location of the target at a sequence of times $t = 1, \ldots, T$, chosen by the user. The target distribution at time t is then the distribution of X_t, and the collection of target paths represents the stochastic process $X = \{X_t, t = 1, \ldots, T\}$. Alternatively this type of target motion can be modeled by using a Markov chain approach as described in Chapter II. The target motion process X then becomes a Markov chain. The target motion models considered in this report and the algorithms discussed below are capable of handling both of these possibilities and indeed a much wider range of target motions. In fact the algorithm in Chapter IV will find optimal allocations for any target motion process which can be represented by a Monte Carlo simulation.

Search grid. For most of the report, we shall assume that a grid has been established on the search space which is two dimensional and that we must allocate our effort uniformly within a given cell of the grid, although effort may vary from cell to cell. Let J denote the set of cells in the grid, and let

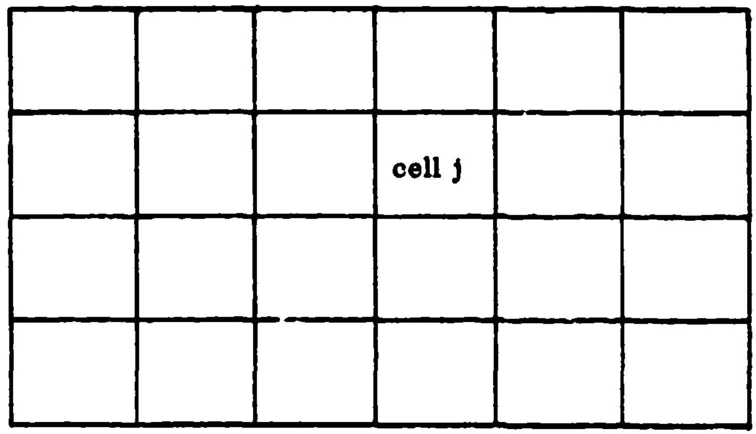
$$\begin{aligned} p_t(j) &= \Pr\{X_t \text{ in cell } j\} \\ &= \Pr\{\text{target is in cell } j \text{ at time } t\} \quad \text{for } j \in J,\ t = 1, \ldots, T, \end{aligned}$$

and

$$A(j) = \text{area of } j^{\text{th}} \text{ cell}, \quad \text{for } j \in J.$$

FIGURE I-1

SEARCH GRID



$p_t(j)$ = Pr{ target is in cell j at time t}

$A(j)$ = area of j^{th} cell

J = collection of cells in the grid

Note: The search grid need not be uniform.

Although we impose this grid structure, the target motion may take place in either discrete or continuous space.

<u>Class of search plans considered.</u> A search plan is described by a nonnegative function ψ of time and space such that

$$\psi(j,t) = \text{effort placed in cell } j \text{ at time } t \text{ for } j \in T,\ t = 1,\ldots,T.$$

We assume that the amount of effort available is constrained so that

$$m(t) = \text{total effort available at time } t.$$

We restrict ourselves to search plans ψ such that

$$\sum_{j \in J} \psi(j,t) = m(t) \quad \text{for } t = 1,\ldots,T. \tag{I-1}$$

The class of plans which satisfy (I-1) is called $\Psi(m)$.

<u>Detection function.</u> Let ω indicate a sample target path drawn from the stochastic process $\{X_t;\ t = 1,\ldots,T\}$. We shall let $X_t(\omega)$ be the target's position (i.e., cell) at time t on this sample path. Suppose that ψ is a search plan. Then the probability of detecting the target given it follows path ω is assumed to be

$$1 - \exp\left(-\sum_{t=1}^{T} W(X_t(\omega))\,\psi(X_t(\omega),t)/A(X_t(\omega))\right) \tag{I-2}$$

where for $j \in J$, $W(j)$ is the sweep width or effectiveness parameter for search in cell j. That is, we are assuming that the detection function is exponential and that detection of the target during one time period is independent of detection during any disjoint time period. In Chapter V, first section, we treat the case where the probability of detection on the path ω is given by $b(z_\omega)$, where z_ω is the effort density which accumulates

on the target along the path ω and b is a regular detection function, i.e., b has a positive continuous and strictly decreasing derivative. When b is not exponential, detection is not independent from time period to time period.

Let E denote expectation over the sample paths of $\{X_t, t = 1, \ldots, T\}$. Suppressing the dependence of X_t on ω, we can write the overall probability of detection by time T as

$$P_T[\psi] \equiv E\left[1-\exp\left(-\sum_{t=1}^{T} W(X_t)\,\psi(X_t, t)/A(X_t)\right)\right] \tag{I-3}$$

for any search plan ψ.

Problem statement. We seek a plan ψ^* which satisfies the effort constraint (I-1) and maximizes the probability of detection by time T within the class of plans satisfying (I-1). Mathematically stated, we seek $\psi^* \in \Psi(m)$ such that

$$P_T[\psi^*] = \max\{P_T[\psi] : \psi \in \Psi(m)\}. \tag{I-4}$$

Such a plan is called T-optimal within $\Psi(m)$.

Basic Necessary and Sufficient Conditions

In this section we present the basic necessary and sufficient conditions which we shall use in constructing the algorithms discussed in this report. The form of the conditions given here applies to discrete-time target motion processes and exponential detection functions. A generalization of these conditions to include continuous time motion processes and non-exponential detection functions is stated and proved in Chapter VI.

In order to state these conditions we let ψ^* denote a plan which is T-optimal within $\Psi(m)$ and let

$$g_{\psi^*}(j,t) = \Pr\left\{\begin{array}{l}\text{target is in cell } j\\ \text{at time } t\end{array}\middle| \begin{array}{l}\text{failure to detect at all times}\\ \text{other than } t \text{ using plan } \psi^*\end{array}\right\} \quad \text{for } j\in J,\ t=1,\ldots,T. \qquad \text{(I-4)}$$

Observe that the conditioning on failure in (I-4) includes search which takes place after time t as well as before. Let E_{jt} denote expectation conditioned on the target being in cell j at time t. Then there is a constant K(t) for $t=1,\ldots,T$ such that

$$g_{\psi^*}(j,t) = \frac{p_t(j)}{K(t)} E_{jt}\left[\exp\left(-\sum_{s\neq t} W(X_t)\,\psi^*(X_t,t)/A(X_t)\right)\right] \quad \text{for } j\in J,\ t=1,\ldots T.$$

The constant K(t) is simply the normalizing constant required so that

$$\sum_{j\in J} g_{\psi^*}(j,t) = 1 \quad \text{for } t=1,\ldots,T.$$

BASIC CONDITION. <u>In order that</u> $\psi^*\in\Psi(m)$ <u>be</u> T-<u>optimal it is necessary and sufficient that the allocation</u> $\psi^*(\cdot,t)$ <u>maximize the probability of detecting a stationary target with distribution</u> $g_{\psi^*}(\cdot,t)$ <u>and effort</u> m(t) <u>for</u> $t=1,\ldots,T$.

To show that if ψ^* is T-optimal within $\Psi(m)$, then it must satisfy the basic condition, we reason as follows: We may write the probability of failing to detect the target by time T using ψ^* as

$$1 - P_T[\psi^*] = E\left[\exp\left(-\sum_{t=1}^{T} W(X_t)\psi^*(X_t, t)/A(X_t)\right)\right]$$

$$= \sum_{j\in J} p_t(j)\, E_{jt}\left[\exp\left(-\sum_{s\neq t} W(X_s)\,\psi^*(X_s, s)/A(X_s)\right)\right] e^{-W(j)\psi^*(j,t)/A(j)} \qquad \text{(I-5)}$$

$$= K(t) \sum_{j\in J} g_{\psi^*}(j, t)\, e^{-W(j)\psi^*(j,t)/A(j)}.$$

One can see from (I-5) that the failure probability $1-P_T[\psi^*]$ for the search over time $[0, T]$ is proportional to the failure probability resulting from applying the allocation $\psi^*(\cdot, t)$ specified by the optimal plan for time t to the stationary target problem with target distribution $g_{\psi^*}(\cdot, t)$. Thus if $\psi^*(\cdot, t)$ did not minimize failure probability (i.e., maximize detection probability) for a stationary target with this distribution under the effort constraint m(t), one could find an allocation f such that $\Sigma_{j\in J} f(j) = m(t)$ and

$$\sum_{j\in J} g_{\psi^*}(j, t) e^{-W(j)\, f(j)/A(j)} < \sum_{j\in J} g_{\psi^*}(j, t) e^{-W(j)\,\psi^*(j,t)/A(j)}.$$

Then one could define

$$\hat{\psi}(j, s) = \begin{cases} \psi^*(j, s) & \text{if } s \neq t \\ f(j) & \text{if } s = t. \end{cases}$$

Clearly the search plan $\hat{\psi}$ is a member of $\Psi(m)$ and $P_T[\Psi] > P_T[\psi^*]$ which contradicts the optimality of ψ^*. Thus $\psi^*(\cdot, t)$ must maximize the probability of detection for a stationary target with distribution $g_{\psi^*}(\cdot, t)$ and effort m(t). This proves the necessity of the basic condition. A proof of the sufficiency is given in Chapter VI.

Description of Basic Algorithm

Most of the algorithms described in this report make use of the basic condition to find a sequence of search plans which converge to the optimal plan. They usually proceed as follows.

First iteration. For time $t = 1$, find the optimal allocation of search effort $m(1)$ for the initial target location distribution p_1. Set $\psi^1(\cdot, 1)$ equal to this allocation and compute g_2^1, the posterior target distribution at time 2 given failure to detect the target at time 1. Set $\psi^1(\cdot, 2)$ equal to the optimal allocation of $m(1)$ effort for the distribution g_2^1. Suppose that we have found $\psi^1(\cdot, s)$ for $s = 1, \ldots, t-1$. Then we continue by computing g_t^1, the posterior target location distribution at time t given failure to detect the target at time $s = 1, \ldots, t-1$, using the allocations $\psi^1(\cdot, s)$ for $s = 1, \ldots, t-1$. Set $\psi^1(\cdot, t)$ equal to the optimal allocation of $m(t)$ effort to the distribution g_t^1. Continue until time T is reached. This constitutes the first pass.

The allocation ψ^1 obtained on this pass is called the incrementally optimal or myopic plan. That is, at each time period ψ^1 allocates its effort in such a way as to maximize the increase in detection probability for that time period. For most moving target problems the myopic plan is not T-optimal for $T > 1$.

Second iteration. Begin this iteration by computing g_1^2, the posterior probability distribution for the target's location at time $t = 1$, given failure to detect the target at times $t = 2, \ldots, T$, using $\psi^1(\cdot, t)$ for the allocation at time $t = 2, \ldots T$. Set $\psi^2(\cdot, 1)$ equal to the optimal allocation of $m(1)$ effort for target distribution g_1^2. Usually $\psi^2(\cdot, 1) \neq \psi^1(\cdot, 1)$, because g_1^2 differs from the initial target distribution, p_1, at time 1. If $\psi^2(\cdot, s)$ has been computed for $s = 1, \ldots, t-1$, then find $\psi^2(\cdot, t)$ as follows: compute

g_t^2, the posterior target location distribution at time t given failure to detect using the allocations $\psi^2(\cdot, s)$ for $s = 1, \ldots, t-1$, and the allocations $\psi^1(\cdot, s)$ for $s = t+1, \ldots, T$, and set $\psi^2(\cdot, t)$ equal to the optimal allocation of m(t) for the distribution g_t^2. Continue until time T is reached. This constitutes the second iteration.

<u>n^{th} iteration.</u> Continue as in the second iteration with ψ^n in place of ψ^2 and ψ^{n-1} in place of ψ^1.

The above algorithm converges to the optimal plan. That is one can find a plan whose probability of detection is as close to optimal as he desires by performing a large enough number of iterations. Notice that each iteration produces a plan with a higher probability of detection than the previous one.

The above algorithm amounts to solving a sequence of stationary target problems to solve an optimal moving target problem. There are very efficient algorithms for solving stationary target problems when the detection function is exponential. (See Example 2.2.8 of reference [h].) Thus we can produce efficient algorithms which follow the procedure outlined above provided we can compute the posterior target distributions g_t^n, $t = 1, \ldots, T$, $n = 1, 2, \ldots$, efficiently. Computation of these distributions will depend on the target motion model, i.e., the nature of the stochastic process $\{X_t, t = 1, \ldots, T\}$. For example, the algorithm described in Chapter II is designed for Markovian target motions. In this case there are very efficient methods for computing the distribution g_t^n. The algorithm described in Chapter IV can be used with any Monte Carlo target motion. This algorithm is more flexible than the one for Markovian motions but it is also slower and requires an additional approximation, i.e., that involved in replacing the stochastic process $\{X_t, t = 1, \ldots, T\}$ by some large but finite number of its sample paths.

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CHAPTER II

MULTISCENARIO MARKOVIAN MOTION

In this chapter* we consider the special case of the basic search problem in which the target moves within a finite number of cells in discrete time. The target's motion process is assumed to be multiscenario Markovian which means that the target is assumed to be following one out of N possible motion scenarios and each scenario is modeled by a discrete time and space, finite Markov chain.

In the first section we discuss the class of searches that will be considered in this chapter. In the second section we present examples of optimal plans. The third section describes the algorithm used to compute the optimal plans and proves that it converges to the optimal plan. The fourth section discusses a method of producing constrained Markovian motions.

Target Motion Model

The target is assumed to be following one of N motion scenarios. The probability that the target is following the n^{th} scenario is α_n where

$$\sum_{n=1}^{N} \alpha_n = 1.$$

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*This chapter is based on reference [b].

Each scenario is modeled by a Markov chain in the following manner. Let Ω be the set of possible target paths over times $t = 1, \ldots, T$. A member ω of Ω specifies the sequence of cells that the target follows from time 1 to T, and

$$\omega = (\omega_1, \ldots, \omega_T),$$

where ω_t is the cell that contains the target at time t. For the n^{th} scenario, $q^n(\omega)$, the probability that target follows path ω, is given by

$$q^n(\omega) = r^n(\omega_1)\, \tau^n(\omega_1, \omega_2)\, \tau^n(\omega_2, \omega_3) \cdots \tau^n_{T-1}(\omega_{T-1}, \omega_T),$$

where $\tau^n_t(i, j)$ is the probability that a target located in cell i at time t will transition to cell j at time t + 1 and $r^n(j)$ is the probability that the target starts in cell j at time 1 under scenario n. Thus each scenario corresponds to a Markov chain with transition probabilities which may vary with time.

The distribution r^n specifies the target distribution at time 1 for scenario n. It is also possible to specify the target's distribution at any additional time $t = 1, \ldots, T$. This has the effect of modifying the transition probabilities τ^n for the scenario. The method of accomplishing this is discussed in the fourth section. It results in path probabilities of the form

$$q^n(\omega) = \rho^n_1(\omega_1)\tau^n_1(\omega_1, \omega_2)\rho^n_2(\omega_2) \cdots \tau^n_{T-1}(\omega_{T-1}, \omega_T)\rho^n_T(\omega_1). \qquad \text{(II-1}$$

Thus, we are considering the search problem described in the first section of Chapter I restricted to the case where the target motion process $\{X_t; t=1, \ldots, T\}$ is a discrete time and space, multiscenario, constrained Markovian process.

Examples

We now give three examples of optimal plans found by the algorithm described in the third section. In all cases J is a subset of a grid of square cells with sides three miles long which are oriented north-south and east-west. One cell is chosen and labeled (0, 0). The cell which is i cells west and j cells north of (0, 0) is labeled (i, j). Each search lasts for eight time intervals of one hour each. We take $W(i, j) = 1$ for all cells (i, j) so that z units of search effort placed in the cell which contains the target will yield a detection with probability $1-e^{-z/9}$.

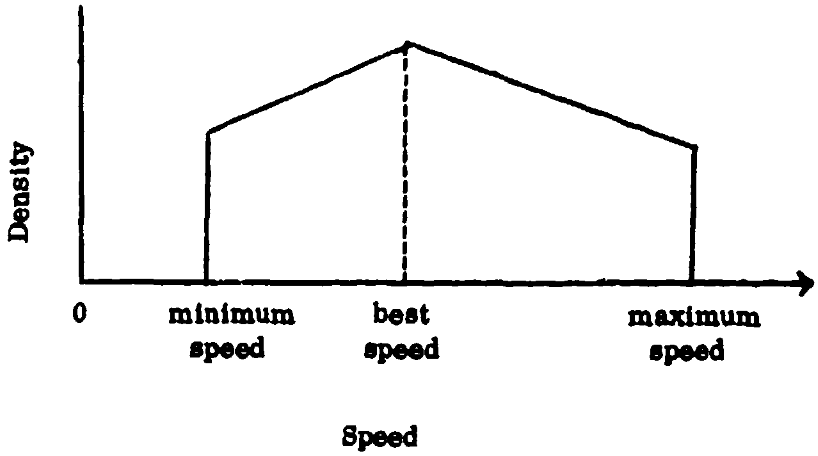
The transition function τ is chosen to model truncated triangular distributions on course and speed. When τ is independent of time, the subscript t will be suppressed. We specify the speed distribution by giving the minimum speed, maximum speed, best speed, and a weight β. The target chooses its speed from the truncated triangular distribution in Figure II-1. The density of the best speed is β times the density of the minimum speed, the latter equals the density of the maximum speed. When $\beta = 1$ this is a uniform distribution. A similar distribution describes the target's course. Let $\tau((i, j), (k, l))$ be the probability that a target which starts at the center of cell (i, j) and which chooses its course and speed from the given triangular distribution moves to cell (k, l) in one hour.

Example 1: Markovian fan. The target's initial distribution is circular normal with a standard deviation of 6 miles. It is centered in the middle of cell (0, 0) and truncated by a 15 mile by 15 mile square whose center coincides with the center of the distribution. The orientation of the square matches the grid. The target's motion is single scenario Markovian with the target choosing its course uniformly from 150° to 210° and its speed between 6 and 12 kts with a best speed of 9 kts, which is given a

FIGURE II-1

TRIANGULAR DISTRIBUTION ON TARGET SPEED

Note: $\beta = 1.5$.



weight of $\beta = 1.5$. Independent draws are made from these distributions each hour. This type of moving target problem is similar to that faced by an aircraft trying to redetect a submarine on which contact has been recently lost.

Table II-1 shows how the target density changes with time in the absence of search. The target moves southward and diffuses from its original distribution. It stays within a sixty degree wedge and has a central tendency because there are more paths which lead to the center of the wedge than to the edges.

We assume $W(j) = 1$ for $j \in J$. Ninety units of search effort are available each hour. Table II-2 gives the final detection probabilities achieved by the first ten iterations of the optimization algorithm described below. The detection probabilities converge very rapidly; indeed the first approximation to the plan, the myopic plan, is quite satisfactory. Table II-3 compares the detection probabilities achieved by the myopic plan at the end of each hour to those achieved by the plans which are optimal for 2 through 8 hours. During the beginning of the search, the myopic plan outperforms the plans which are optimal for later times. In no case is the myopic plan significantly outperformed by any optimal plan.

Table II-4 shows the myopic plan and the optimal plan for eight hours oriented with north at the top of the page. The myopic plan starts by putting most of its effort in the center of the target distribution. It diffuses and moves southward with the target. The optimal plan, on the other hand, starts by using most of its effort to surround the target. By the end of the search the optimal plan closes in on the target. At hour eight the optimal plan puts 70 percent more effort in the center of the target's distribution than the myopic plan does.

TABLE II-1

TARGET DENSITY FOR THE MARKOVIAN FAN

Notes: (1) Entries represent thousandths of a target.
(2) Cells are 3 mi x 3 mi.

Hour 1

	2	1	0	-1	-2
2]	23	34	33	34	23
1]	34	49	50	49	34
0]	33	56	63	56	33
-1]	34	49	50	49	34
-2]	23	31	33	34	23

Hour 2

	4	3	2	1	0	-1	-2	-3	-4
0]	0	2	6	10	11	10	6	2	0
-1]	1	5	19	30	34	30	19	5	1
-2]	1	12	25	41	50	44	28	12	1
-3]	2	13	30	43	54	48	30	13	2
-4]	1	11	26	41	46	41	26	11	1
-5]	1	6	15	23	26	23	15	6	1
-6]	0	1	3	5	5	5	3	1	0

Hour 3

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-2]	0	0	1	2	3	3	3	2	1	0	0
-3]	0	1	4	9	13	15	13	9	4	1	0
-4]	1	1	10	20	23	32	28	20	10	4	1
-5]	1	5	14	27	39	44	39	27	14	5	1
-6]	1	5	15	23	40	46	40	23	15	5	1
-7]	1	4	12	23	33	37	33	23	12	4	1
-8]	1	3	7	13	18	20	18	13	7	3	1
-9]	0	1	2	4	5	6	5	4	2	1	0
-10]	0	0	0	0	1	1	1	0	0	0	0

TABLE 1 (Continued)

Hour 4

	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6
-4]	0	0	0	0	1	1	1	1	1	0	0	0	0
-5]	0	0	1	2	4	5	6	5	4	2	1	0	0
-6]	0	1	3	6	11	15	16	15	11	6	3	1	0
-7]	0	2	5	12	20	27	30	27	20	12	5	2	0
-8]	0	2	7	15	26	35	38	35	26	15	7	2	0
-9]	1	2	7	16	26	35	38	35	25	16	7	2	1
-10]	0	2	6	12	20	27	30	27	20	12	6	2	0
-11]	0	1	3	7	11	15	17	15	11	7	3	1	0
-12]	0	0	1	2	4	5	6	5	4	2	1	0	0
-13]	0	0	0	0	1	1	1	1	1	0	0	0	0

Hour 5

	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6
-7]	0	0	0	1	1	2	2	2	1	1	0	0	0
-8]	0	1	2	3	5	7	7	7	5	3	2	1	0
-9]	0	2	4	8	12	16	17	16	12	8	4	2	0
-10]	1	3	7	13	20	26	28	26	20	13	7	3	1
-11]	1	4	8	16	24	31	34	31	24	16	8	4	1
-12]	1	4	8	15	24	30	33	30	24	15	8	4	1
-13]	1	3	6	12	18	23	25	23	18	12	6	3	1
-14]	1	2	4	7	10	13	14	13	10	7	4	2	1
-15]	0	1	1	3	4	5	5	5	4	3	1	1	0
-16]	0	0	0	1	1	1	1	1	1	1	0	0	0

Hour 6

	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7
-9]	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0
-10]	0	0	0	1	2	2	3	3	3	2	2	1	0	0	0
-11]	0	0	1	3	4	7	8	9	8	7	4	3	1	0	0
-12]	0	1	2	5	9	13	16	18	16	13	9	5	2	1	0
-13]	0	1	4	8	13	19	24	26	24	19	13	8	4	1	0
-14]	1	2	4	9	16	23	28	30	28	23	16	9	4	2	1
-15]	1	2	4	9	15	21	26	28	26	21	15	9	4	2	1
-16]	0	1	3	5	11	16	20	21	20	16	11	6	3	1	0
-17]	0	1	2	4	6	9	11	12	11	9	6	4	2	1	0
-18]	0	0	1	2	3	4	5	5	5	4	3	2	1	0	0
-19]	0	0	0	0	1	1	1	2	1	1	1	0	0	0	0

TABLE 1 (Continued)

Hour 7

	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7
-12	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0
-13	0	0	1	1	2	3	4	4	4	3	2	1	1	0	0
-14	0	1	2	3	5	8	9	10	9	8	5	3	2	1	0
-15	1	1	3	6	10	14	17	18	17	14	10	6	3	1	1
-16	1	2	5	8	14	19	23	25	23	19	14	8	5	2	1
-17	1	2	5	10	15	21	26	27	26	21	15	10	5	2	1
-18	1	2	5	9	14	19	23	25	23	19	14	9	5	2	1
-19	1	2	4	6	10	14	17	18	17	14	10	6	4	2	1
-20	0	1	2	4	6	8	10	11	10	8	6	4	2	1	0
-21	0	0	1	2	3	4	4	5	4	4	3	2	1	0	0
-22	0	0	0	1	1	1	1	2	1	1	1	1	0	0	0

Hour 8

	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8
-14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-15	0	0	0	0	1	1	1	2	2	2	1	1	1	0	0	0	0
-16	0	0	1	1	2	3	4	5	5	5	4	3	2	1	1	0	0
-17	0	0	1	2	4	6	9	10	11	10	9	6	4	2	1	0	0
-18	0	1	2	4	7	10	14	17	18	17	14	10	7	4	2	1	0
-19	0	1	3	5	9	14	18	22	23	22	18	14	9	5	3	1	0
-20	0	1	3	5	10	15	20	23	25	23	20	15	10	6	3	1	0
-21	0	1	3	5	9	13	17	21	22	21	17	13	9	5	3	1	0
-22	0	1	2	4	6	10	13	15	16	15	13	10	6	4	2	1	0
-23	0	1	1	2	4	6	7	9	9	9	7	6	4	2	1	1	0
-24	0	0	1	1	2	3	3	4	4	4	3	3	2	1	1	0	0
-25	0	0	0	0	1	1	1	1	2	1	1	1	1	0	0	0	0
-26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE II-2

CONVERGENCE OF FINAL DETECTION PROBABILITIES FOR THE MARKOVIAN FAN

Number of Iterations	Probability of Detection After 8 Hours
1	.803
2	.818
3	.818
4	.818
5	.818
6	.818
7	.818
8	.818
9	.818
10	.818

TABLE II-3

DETECTION PROBABILITIES FOR THE MARKOVIAN FAN

Number of hours	Myopic plan	Plan which is optimal for 2 hours	3 hours	4 hours	5 hours	6 hours	7 hours	8 hours
1	.357	.348	.336	.328	.322	.317	.314	.311
2	.531	.535	.531	.526	.522	.519	.517	.515
3	.627		.636	.634	.631	.628	.626	.624
4	.687			.699	.697	.695	.693	.692
5	.730				.743	.742	.740	.739
6	.760					.775	.774	.773
7	.784						.799	.798
8	.803							.818

TABLE II-4

SEARCH PLANS FOR THE MARKOVIAN FAN

Notes: (1)Entries represent thousandths of a unit of search density (effort/mi^2).
(2)Ninety units of search effort are available each hour.
(3)Cells are 3 mi x 3 mi.

Myopic Plan

N ↑

Hour 1

	2	1	0	-1	-2
2]	0	255	381	256	0
1]	256	631	756	631	256
0]	381	756	831	756	381
-1]	256	631	756	631	256
-2]	0	256	381	256	0

Hour 2

	4	3	2	1	0	-1	-2	-3	-4
0]	0	0	0	0	0	0	0	0	0
-1]	0	0	78	469	525	469	78	0	0
-2]	0	0	269	646	692	646	269	0	0
-3]	0	0	279	650	693	650	279	0	0
-4]	0	0	262	639	693	639	262	0	0
-5]	0	0	0	246	323	246	0	0	0
-6]	0	0	0	0	0	0	0	0	0

Hour 3

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-2]	0	0	0	0	0	0	0	0	0	0	0
-3]	0	0	0	0	2	55	2	0	0	0	0
-4]	0	0	0	288	457	484	457	288	0	0	0
-5]	0	0	0	397	534	548	534	397	0	0	0
-6]	0	0	3	401	535	548	535	401	3	0	0
-7]	0	0	0	358	522	548	522	358	0	0	0
-8]	0	0	0	28	239	293	239	28	0	0	0
-9]	0	0	0	0	0	0	0	0	0	0	0
-10]	0	0	0	0	0	0	0	0	0	0	0

TABLE II-4 (continued)

Hour 4

	4	3	2	1	0	-1	-2	-3	-4
-5	0	0	0	0	0	0	0	0	0
-6	0	0	69	222	268	222	69	0	0
-7	0	60	311	381	395	381	311	60	0
-8	0	159	366	396	397	396	366	159	0
-9	0	162	367	397	397	397	367	162	0
-10	0	84	325	395	397	395	325	84	0
-11	0	0	64	199	228	199	64	0	0
-12	0	0	0	0	0	0	0	0	0

Hour 5

	4	3	2	1	0	-1	-2	-3	-4
-8	0	0	0	0	0	0	0	0	0
-9	0	0	188	263	280	263	188	0	0
-10	0	172	309	334	335	334	309	172	0
-11	0	220	324	335	335	335	324	220	0
-12	0	215	325	335	335	335	325	215	0
-13	0	146	299	335	335	335	299	146	0
-14	0	0	94	189	210	189	94	0	0
-15	0	0	0	0	0	0	0	0	0

Hour 6

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-13	0	0	0	0	0	0	0	0	0	0	0
-14	0	0	0	0	77	109	77	0	0	0	0
-15	0	0	104	228	262	269	262	228	104	0	0
-16	0	2	214	277	285	285	285	277	214	2	0
-17	0	55	239	282	285	285	285	282	239	55	0
-18	0	38	230	282	285	285	285	282	230	38	0
-19	0	0	169	265	285	285	285	265	169	0	0
-20	0	0	0	106	176	192	176	106	0	0	0
-21	0	0	0	0	0	0	0	0	0	0	0

TABLE II-4 (continued)

Hour 7

	5	4	3	2	1	0	-1	-2	-3	-4	-3
-13	0	0	0	0	0	0	0	0	0	0	0
-14	0	0	0	100	162	177	162	100	0	0	0
-15	0	0	160	223	239	241	239	223	160	0	0
-16	0	91	220	241	242	242	242	241	220	91	0
-17	0	119	229	242	242	242	242	242	229	119	0
-18	0	97	221	242	242	242	242	242	221	97	0
-19	0	0	173	233	242	242	242	233	173	0	0
-20	0	0	0	107	159	171	159	107	0	0	0
-21	0	0	0	0	0	0	0	0	0	0	0

Hour 8

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-16	0	0	0	0	0	0	0	0	0	0	0
-17	0	0	63	149	178	185	178	149	63	0	0
-18	0	65	182	211	215	215	215	211	182	65	0
-19	0	132	205	215	215	215	215	215	205	132	0
-20	0	145	209	215	215	215	215	215	209	145	0
-21	0	127	208	215	215	215	215	215	208	127	0
-22	0	46	175	212	215	215	215	212	175	46	0
-23	0	0	7	110	149	159	149	110	7	0	0
-24	0	0	0	0	0	0	0	0	0	0	0

TABLE II-4 (continued)

Optimal Plan for 8 Hours

Hour 1

	2	1	0	-1	-2
2]	424	544	559	544	424
1]	422	327	235	327	422
0]	416	259	173	259	416
-1]	427	352	230	352	427
-2]	411	520	526	520	411

Hour 2

	3	2	1	0	-1	-2	-3
0]	0	0	0	0	0	0	0
-1]	0	240	437	496	437	240	0
-2]	0	370	533	547	533	370	0
-3]	0	363	527	545	527	363	0
-4]	0	335	514	526	514	335	0
-5]	0	74	320	359	320	74	0
-6]	0	0	0	0	0	0	0

Hour 3

	4	3	2	1	0	-1	-2	-3	-4
-2]	0	0	0	0	0	0	0	0	0
-3]	0	0	0	106	151	106	0	0	0
-4]	0	0	285	343	355	343	285	0	0
-5]	0	153	367	411	429	411	367	153	0
-6]	0	183	382	430	447	430	382	183	0
-7]	0	117	351	411	420	411	351	117	0
-8]	0	0	116	267	292	267	146	0	0
-9]	0	0	0	0	0	0	0	0	0
-10]	0	0	0	0	0	0	0	0	0

TABLE II-4 (continued)

Hour 4

	4	3	2	1	0	-1	-2	-3	-4
-5	0	0	0	0	0	0	0	0	0
-6	0	0	158	255	263	255	158	0	0
-7	0	167	300	323	334	323	300	167	0
-8	0	233	315	343	356	343	315	233	0
-9	0	229	310	336	343	336	310	229	0
-10	0	172	284	308	314	308	284	172	0
-11	0	0	134	205	220	205	134	0	0
-12	0	0	0	0	0	0	0	0	0

Hour 5

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-8	0	0	0	0	0	0	0	0	0	0	0
-9	0	0	48	194	238	243	238	194	48	0	0
-10	0	0	204	271	297	303	297	271	204	0	0
-11	0	51	244	293	327	340	327	293	244	51	0
-12	0	47	238	290	321	331	321	290	238	47	0
-13	0	0	189	265	288	295	288	265	189	0	0
-14	0	0	0	136	194	207	194	136	0	0	0
-15	0	0	0	0	0	0	0	0	0	0	0

Hour 6

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-10	0	0	0	0	0	0	0	0	0	0	0
-11	0	0	0	0	75	101	75	0	0	0	0
-12	0	0	116	204	230	234	230	204	116	0	0
-13	0	53	213	253	274	283	274	253	213	53	0
-14	0	113	239	271	296	305	296	271	239	113	0
-15	0	96	229	262	286	295	286	262	229	96	0
-16	0	0	178	235	254	261	254	235	178	0	0
-17	0	0	0	115	167	179	167	115	0	0	0
-18	0	0	0	0	0	0	0	0	0	0	0

TABLE II-4 (continued)

Hour 7

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-161	0	0	0	0	0	0	0	0	0	0	0
-171	0	0	0	53	115	133	115	53	0	0	0
-181	0	0	122	209	256	271	256	209	122	0	0
-191	0	66	206	283	330	345	330	283	206	66	0
-201	0	98	230	305	353	368	353	305	230	98	0
-211	0	63	204	276	320	335	320	276	204	63	0
-221	0	0	111	196	237	249	237	196	111	0	0
-231	0	0	0	4	57	36	67	4	0	0	0
-241	0	0	0	0	0	0	0	0	0	0	0

Hour 8

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-161	0	0	0	0	0	0	0	0	0	0	0
-171	0	0	0	53	115	153	115	53	0	0	0
-181	0	0	122	209	256	271	256	209	122	0	0
-191	0	66	206	283	330	346	330	283	206	66	0
-201	0	98	230	305	353	368	353	305	230	98	0
-211	0	63	204	276	320	336	320	276	204	63	0
-221	0	0	111	196	237	249	237	196	111	0	0
-231	0	0	0	4	57	36	67	4	0	0	0
-241	0	0	0	0	0	0	0	0	0	0	0

Example 2: Constrained Markovian fan. In this example we consider a target motion which is identical to the one in Example 1 except that the target is constrained to have a truncated normal distribution at time 8 which is equal to the initial distribution translated 60 miles due south.

The transition function τ for this constrained process is computed by the method discussed in the fourth section.

Table II-5 shows how the target density changes over the first four hours in the absence of search. The target moves southward and diffuses from its original distribution. During the last four hours the target density is a mirror image of the first four hours. It still moves south but converges back to a truncated normal distribution with a 6 mile standard deviation.

Ninety units of search effort are available each hour. The detection probabilities for successive iterations of the algorithm are .926, .932, .932, .932, and .932 which repeats. These probabilities converge rapidly and again the first approximation to the plan, the myopic plan, is quite satisfactory. Table II-6 compares the detection probabilities achieved by the myopic plan at the end of each hour to those achieved by the plans which are optimal for 2, 4, 6, and 8 hours. Again, no optimal plan significantly outperforms the myopic plan. With the exception of Example 3 below, these conclusions have been confirmed by other examples including some with much smaller overall detection probabilities.

Table II-7 shows the optimal plan for eight hours. Again, only the tables for the first four hours are displayed since the last four hours are a mirror image of them. At hour 1, the optimal plan uses most of its effort to surround the target rather than searching the highest probability cells as the myopic plan, which is shown at the

TABLE II-5

TARGET DENSITY FOR THE CONSTRAINED MARKOVIAN FAN

Notes: (1) Entries represent thousandths of a target.
(2) Cells are 3 mi by 3 mi.

N

Hour 1

	2	1	0	-1	-2
2]	23	34	38	34	23
1]	34	49	56	49	34
0]	38	56	63	56	38
-1]	34	49	56	49	34
-2]	23	34	38	34	23

Hour 2

	4	3	2	1	0	-1	-2	-3	-4
0]	0	1	3	7	7	7	3	1	0
-1]	0	6	17	31	36	31	17	6	0
-2]	1	9	27	50	58	50	27	9	1
-3]	1	10	30	55	64	55	30	10	1
-4]	1	9	25	47	54	47	25	9	1
-5]	0	4	12	23	26	23	12	4	0
-6]	0	0	2	3	3	3	2	0	0

Hour 3

	4	3	2	1	0	-1	-2	-3	-4
-2]	0	0	1	1	1	1	1	0	0
-3]	0	2	5	9	11	9	5	2	0
-4]	2	7	18	31	37	31	18	7	2
-5]	2	11	29	49	59	49	29	11	2
-6]	3	12	31	53	63	53	31	12	3
-7]	2	9	24	40	48	40	24	9	2
-8]	1	4	10	17	21	17	10	4	1
-9]	0	1	2	3	3	3	2	1	0

TABLE II-5 (continued)

Hour 4

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-5]	0	0	0	1	2	2	2	1	0	0	0
-6]	0	1	3	7	12	14	12	7	3	1	0
-7]	0	2	8	19	33	39	33	19	8	2	0
-8]	0	3	12	30	50	60	50	30	12	3	0
-9]	0	3	12	31	52	61	52	31	12	3	0
-10]	0	2	9	21	36	43	36	21	9	2	0
-11]	0	1	3	8	14	17	14	8	3	1	0
-12]	0	0	1	1	2	3	2	1	1	0	0

Hours 5-8 are a mirror reflection of hours 1-4.

TABLE II-6

DETECTION PROBABILITIES FOR THE CONSTRAINED MARKOVIAN FAN

Number of hours	Myopic plan	Plan which is optimal for 2 hours	4 hours	6 hours	8 hours
1	.357	.346	.324	.313	.311
2	.555	.560	.549	.541	.540
3	.672		.679	.673	.672
4	.750		.762	.758	.757
5	.807			.818	.817
6	.852			.862	.862
7	.889				.898
8	.926				.931

TABLE II-7

SEARCH PLANS FOR THE CONSTRAINED MARKOVIAN FAN

Notes: (1) Entries represent thousandths of a unit of search density (effort/mi^2)
(2) Cells are 3 mi by 3 mi.

Optimal Plan for 8 Hours

Hour 1

N

	2	1	0	-1	-2
2]	448	563	568	563	448
1]	423	361	263	361	423
0]	392	268	165	268	392
-1]	403	338	262	338	403
-2]	416	508	502	508	416

Hour 2

	4	3	2	1	0	-1	-2	-3	-4
0]	0	0	0	0	0	0	0	0	0
-1]	0	0	101	485	532	485	101	0	0
-2]	0	0	338	605	616	605	338	0	0
-3]	0	0	363	602	617	602	363	0	0
-4]	0	0	312	583	599	583	312	0	0
-5]	0	0	0	268	323	268	0	0	0
-6]	0	0	0	0	0	0	0	0	0

Hour 3

	4	3	2	1	0	-1	-2	-3	-4
-2]	0	0	0	0	0	0	0	0	0
-3]	0	0	0	0	0	0	0	0	0
-4]	0	0	257	449	484	449	257	0	0
-5]	0	0	424	546	568	546	424	0	0
-6]	0	0	435	553	580	553	435	0	0
-7]	0	0	363	518	544	518	363	0	0
-8]	0	0	0	223	285	223	0	0	0
-9]	0	0	0	0	0	0	0	0	0

TABLE II-7 (continued)

Hour 4

	5	4	3	2	1	0	-1	-2	-3	-4	-5
-5]	0	0	0	0	0	0	0	0	0	0	0
-6]	0	0	0	0	52	121	52	0	0	0	0
-7]	0	0	0	302	430	452	430	302	0	0	0
-8]	0	0	74	436	503	523	503	436	74	0	0
-9]	0	0	84	441	512	534	512	441	84	0	0
-10]	0	0	0	343	469	492	469	343	0	0	0
-11]	0	0	0	0	173	240	173	0	0	0	0
-12]	0	0	0	0	0	0	0	0	0	0	0

Hours 5-8 are a mirror reflection of hours 1-4.

Myopic Plan

Hour 1

1]	2	1	0	-1	-2
2]	0	256	381	256	0
1]	256	631	756	631	256
0]	381	756	881	756	381
-1]	256	631	756	631	256
-2]	0	256	381	256	0

Hours 2-3 not shown.

Hour 4

4]	5	4	3	2	1	0	-1	-2	-3	-4	-5
-5]	0	0	0	0	0	0	0	0	0	0	0
-6]	0	0	0	0	0	26	0	0	0	0	0
-7]	0	0	0	242	505	556	505	242	0	0	0
-8]	0	0	0	402	592	614	592	402	0	0	0
-9]	0	0	0	401	590	607	590	401	0	0	0
-10]	0	0	0	295	566	626	566	295	0	0	0
-11]	0	0	0	0	92	205	92	0	0	0	0
-12]	0	0	0	0	0	0	0	0	0	0	0

Hours 5-8 not shown.

end of Table II-7, does. By hour 4, however, the optimal plan concentrates its effort in the center of the target distribution.

Example 3: Multiple scenario motion. Next we consider a multiscenario example. Figure II-2 illustrates the two possible scenarios for target motion with (i, j) indicating the midpoint of cell (i, j). In scenario 1 the target starts uniformly distributed over the square in Figure II-2 which is northwest of the obstruction. It proceeds southward to distribute its density at hour 4 uniformly over the square southwest of the obstruction. Within the limits imposed by these constraints, the target chooses courses and speeds from the triangular distribution given in row I of the table at the bottom of Figure II-2. After hour 4 the range of courses and speeds widens as given in row II. In scenario 2 the target again starts in the northwest rectangle but now it goes around the obstruction to the north and east. Uniform distributions over the squares shown in Figure 2 are imposed on the target location distribution at hours 1, 4, and 7. The course and speed distributions that hold between hours 1-3, 4-6, and 7-8 are given by rows III, IV, and V.

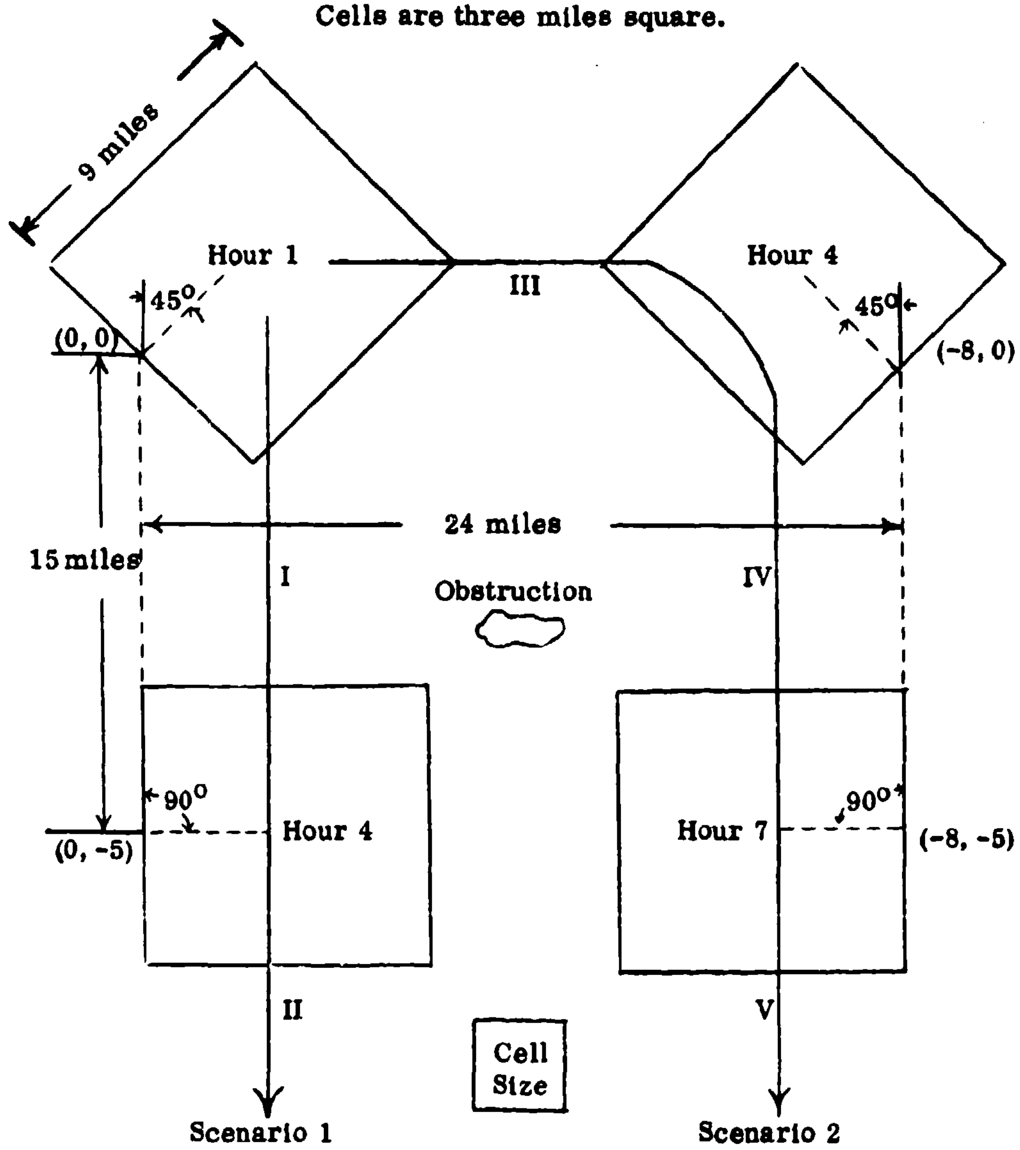
Search effort becomes available gradually during the eight hours of the search. No search is available for the first two hours. Nine units of search effort are available during each of hours 3 and 4 and 36 units of search effort are available during each of hours 5 through 8. The final detection probabilities obtained by successive iterations of the algorithm are .465, .519, .519, .519, and .519 which repeats. After a jump between the first and second iteration, the convergence is characteristically rapid.

Table II-8 compares the detection probabilites for the myopic plan with the optimal plans for 4 through 8 hours. For 8 hours the optimal plan improves the detection probabilities from .465 to .519. The qualitative difference in the plans

FIGURE II-2

GEOGRAPHY FOR MULTISCENARIO MOTION

Note: Staging areas are nine miles square.
Cells are three miles square.



	Course min.	Course max.	Course best	Course weight	Speed (knots) min.	Speed (knots) max.	Speed (knots) best	Speed (knots) weight
I	160°	200°	180°	2	3	9	6	2
II	110°	250°	180°	1	6	12	9	1.5
III	70°	110°	90°	2	3	9	6	2
IV	160°	200°	180°	2	3	9	6	2
V	150°	210°	180°	2	3	9	6	2

TABLE II-8

DETECTION PROBABILITIES FOR THE MULTISCENARIO MOTION

Number of hours	Myopic plan	Plan which is optimal for 4 hours	5 hours	6 hours	7 hours	8 hours
3	.063	.062	.058	.041	.042	.042
4	.112	.112	.097	.078	.078	.078
5	.256		.269	.262	.262	.252
6	.351			.382	.380	.376
7	.425					.519
8	.465					

which causes this increase is shown in Table II-9. Because the second scenario has higher weight, the myopic plan searches it almost exclusively during the early hours. During the later hours, the first scenario diffuses so widely that the second scenario is still searched almost exclusively. The myopic plan is handicapped because it does not look ahead to see that first scenario targets get away quickly and search this scenario while it is still possible. The optimal plan does this. It searches the first scenario until it starts to diffuse, then shifts to the second scenario. This strategy is optimal for the long term but, as Table II-8 shows, involves a penalty during hours 3 and 4.

Further examples of optimal search plans computed by these techniques can be found in reference [a].

Description of Algorithm

Recall from the discussion in the third section of Chapter I that the basic step in the optimization algorithm is a reallocation of search effort at a single time interval. Example 2.2.8 of reference [h] gives an algorithm for finding an optimal allocation of search effort for a stationary target in a discrete search space when the detection function is exponential. Using the necessary conditions in Corollary 2.1.6 of reference [h] one can show that the optimal allocation is unique.

Suppose that we have a search plan ψ. Let $g_\psi(\cdot, t)$ be the posterior target location distribution at time t given failure to detect at all times other than t using the plan ψ. Let f^* be the optimal allocation of m(t) effort for the stationary target problem with distribution $g_\psi(\cdot, t)$. Define $\Xi_t\psi$ to be the search plan that is obtained from ψ by replacing the allocation at time t with the allocation f^*. That is, for $j \in J$,

$$\Xi\, \psi(j, s) = \begin{cases} \psi(j, s) & \text{for } s \neq t \\ f^*(j) & \text{for } s = t. \end{cases}$$

TABLE II-9

SEARCH PLANS FOR MULTISCENARIO MOTION

Notes: (1) Entries represent thousandths of a unit of search density (effort/mi^2).
(2) ○ indicates the position of the obstruction.
(3) Cells are 3 mi x 3 mi.

N ↑

Hour 1 -- No Search Available

Hour 2 -- No Search Available

Hour 3

Myopic Plan

0	0	0	0	286	251
0	0	0	0	246	217
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Optimal plan for 8 hours

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
204	281	0	0	0	0
211	304	0	0	0	0

Hour 4

Myopic Plan

0	0	0	0	0	0	46	0
0	0	0	0	0	0	202	231
0	0	0	0	0	0	223	245
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
9	9	9	0	0	0	0	0
9	9	9	0	0	0	0	0

Optimal plan for 8 hours

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
154	92	266	0	0	0	0	0
157	94	237	0	0	0	0	0

TABLE II-9 (continued)

Myopic Plan									Optimal plan for 8 hours								
Hour 5																	
0	0	0	0	0	0	0	241	0	0	0	0	0	0	0	0	324	112
0	0	0	0	0	0	464	716	675	0	0	0	0	0	0	287	645	569
0	0	0	0	0	0	321	714	730	0	0	0	0	0	0	165	654	563
0	0	0	0	0	0	0	140	0	0	0	0	0	0	0	0	217	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	24	150	111	151	24	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0

Hour 6

Myopic Plan					Optimal plan for 8 hours				
0	0	0	201	94	0	0	0	279	130
0	0	526	548	627	0	0	395	631	581
0	0	471	556	682	0	0	345	657	596
0	0	0	161	133	0	0	0	279	107

Hour 7

Myopic Plan					Optimal plan for 8 hours				
0	0	261	82	214	0	0	353	0	248
0	0	537	316	556	0	0	611	79	538
0	0	525	312	594	0	0	614	110	603
0	0	224	114	264	0	0	356	90	398

Hour 8

Myopic Plan					Optimal plan for 8 hours				
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	203	428	204	0	0	152	427	153
0	0	376	609	376	0	0	420	711	421
0	0	376	609	377	0	0	392	674	392
0	0	59	326	58	0	0	0	259	0

The algorithm proceeds as follows:

(1) Let ψ_0 be an initial guess for the search plan.

(2) Let ϵ be a small number.

(3) Let $l = 0$.

(4) Perform step (5) for t = 1 to T.

(5) Set $\psi_{Tl+t} = \Xi_t \psi_{Tl+t-1}$.

(6) If $| P_T[\psi_{(l+1)T}] - P_T[\psi_{lT}] | < \epsilon$, stop: the answer is $\psi_{(l+1)T}$.

(7) Otherwise increase l by 1, and return to step (4).

While the initial guess ψ_0 may be any search plan, we generally use $\psi_0(j, t) = 0$ for $j \in J$, $t = 1, \ldots, T$. In this case forming $\psi_1, \psi_2, \ldots, \psi_T$ corresponds to allocating search effort at successive time intervals in order to obtain the greatest increase in P_T at the current time interval -- not to maximize P_T in the long term. Thus, ψ_T is the myopic or incrementally optimal plan.

Implementation of the algorithm. The only difficult step in the algorithm is the computation of Ξ_t. Let $\psi = \psi_{lT+t-1}$. Then this step consists of computing the distribution $g_\psi(\cdot, t)$ and finding an optimal allocation of m(t) effort for this distribution when the detection function is exponential. Example 2.2.8 of reference [h] gives an algorithm for finding such an allocation, so we consider only the computation of $g_\psi(\cdot, t)$.

Since the algorithm for finding an optimal allocation for $g_\psi(\cdot, t)$ produces the same allocation for any "distribution" $Kg_\psi(\cdot, t)$, where K is a positive constant, we shall be concerned only with calculating g_ψ in an unnormalized form.

For $j \in J$, define

$$R(j, 1, \psi) = \rho_1(j),$$

and for $1 < t \leq T$, let

$$R(j, t, \psi) = \sum_{k \in J} R(k, t-1, \psi) \exp(-W(k)\psi(k, t-1)/A(k))\tau_{t-1}(k, j)\rho_t(j).$$

Similarly for $j \in J$, let

$$S(j, T, \psi) = 1$$

and for $1 \leq t < T$,

$$S(j, t, \psi) = \sum_{k \in J} \tau_t(j, k)\rho_{t+1}(k) \exp(-W(k)\psi(k, t+1)/A(k))S(k, t+1, \psi).$$

If ρ_1 is the initial distribution and $\rho_t(j) = 1$ for $t>1$ and $j \in J$, then R and S have natural probability interpretations. In fact, $R(j, t, \psi)$ is the probability that the target reaches cell j at time t and is not detected by the effort at times $s = 1, \ldots, t-1$ while $S(j, t, \psi)$ is the probability that if the target starts in cell j at time t it will not be detected by the effort at times $s = t+1, \ldots, T$. As a result, $R(j, t, \psi)S(j, t, \psi)$ is the probability that the target is located in cell j at time t and is not detected by the effort at all times other than t. It follows that

$$g_\psi(j, t) = R(j, t, \psi)S(j, t, \psi)/K(t) \quad \text{for} \quad j \in J, \ 1 \leq t \leq T$$

where

$$K(t) = \sum_{j \in J} R(j, t, \psi)S(j, t, \psi) \qquad 1 \leq t \leq T.$$

In the fourth section it is shown that up to a constant factor the probabilistic interpretation

of R and S remains true for the weights ρ constructed for the constrained Markovian motion. Thus, $g_\psi(\cdot, t)$ may be computed by the above formula for constrained Markov motion. As we noted before, the algorithm in Example 2.2.8 of reference [h] is insensitive to multiplication of the target distribution by a constant, so the factor K(t) is not computed.

Observe that it is not necessary to recompute the entire R and S functions at each step in optimization. Suppose one has reached time T in the algorithm and is about to begin at time 1 again, i.e., one has returned to step (4) in the algorithm description. The allocation just computed is ψ_{lT} for some l. At this point one computes $S(\cdot, t, \psi_{lT})$ recursively for $t = 1, \ldots, T-1$, calculates $g_{\psi_{lT}}(\cdot, 1)$ from (II-4) and reallocates effort to obtain ψ_{lT+1}. Having done this one computes $R(\cdot, 2, \psi_{lT+1})$ from (II-2) and then $g_{\psi_{lT+1}}$ from (II-4) using the fact that $S(\cdot, 2, \psi_{lT+1}) = S(\cdot, 2, \psi_{lT})$ because the reallocation of effort at time 1 does not affect the computation of $S(\cdot, 2, \psi_{lT+1})$. The distribution $g_{\psi_{lT+t}}$ is computed in a similar fashion for $t = 2, \ldots, T$. Thus for one cycle through the time periods $t = 1, \ldots, T$, one need compute R and S only once.

If there is more than one scenario, then one proceeds as above to compute $g^n_\psi(\cdot, t)$ for the n^{th} scenario and takes $g_\psi(\cdot, t) = \sum_{n=1}^{N} \alpha_n g^n_\psi(\cdot, t)$.

The computer program which implements this algorithm is described in references [l] and [m].

<u>Convergence of the algorithm.</u> We now show that this algorithm halts, that the resulting plan $\psi^{\#}$ is in $\Psi(m)$ and that as ϵ approaches zero, $\psi^{\#}$ approaches optimality. The definition of Ξ_t shows that $\psi^{\#} \in \Psi(m)$ and that $0 \le P_T[\psi_0] \le P_T[\psi_1] \le \cdots \le 1$. Thus $\lim_{l\to\infty} P_T[\psi_l]$ exists and $\lim_{l\to\infty} \left| P_T[\psi_{(l+1)T}] - P_T[\psi_{lT}] \right| = 0$.

It follows that the algorithm halts. As $\epsilon \to 0$, the answer is farther into the sequence $\psi_1, \psi_2, \ldots$, so we need only show that $\lim_{l\to\infty} P_T[\psi_l] = \max\{P_T[\psi] : \psi \in \Psi(m)\}$. In fact, we shall show that for some subsequence $\{l_k\}_{k=1}^{\infty}$, $\psi^* = \lim_{k\to\infty} \psi_{l_k}$ exists and is T-optimal within $\Psi(m)$.

Since J and T are finite and $m(t)$, $t = 1, \ldots, T$ is bounded, we may consider ψ_l, $l = 1, 2, \ldots$ to be a sequence of points in a bounded set of a finite dimensional Euclidean space. Thus, there is a subsequence $\{l_k\}_{k=1}^{\infty}$ such that $\{\psi_{l_k} : k=1, 2, \ldots\}$ converges to a point ψ^*. Clearly $\psi^* \in \Psi(m)$. The continuity of P_T implies that $\lim_{l\to\infty} P_T[\psi_l] = \lim_{k\to\infty} P_T[\psi_{l_k}] = P_T[\psi^*]$ so that we need only show that $P[\psi^*] = \max\{P_T[\psi] : \psi \in \Psi(m)\}$, i.e., ψ^* is T-optimal within $\Psi(m)$.

Since $\lim_{l\to\infty} P_T[\psi_l]$ exists and P_T and Ξ_t are continuous for $t = 1, \ldots, T$, it follows that

$$P_T[\psi^*] = \lim_{k\to\infty} P_T[\psi_{l_k}] = \lim_{k\to\infty} P_T[\psi_{l_k+1}]$$

$$= \lim_{k\to\infty} P_T[\Xi_1(\psi_{l_k T})] = P_T[\Xi_1(\psi^*)].$$

Since $P_T[\psi^*] = P_T[\Xi_1\psi^*]$, it follows that both $\psi^*(\cdot, 1)$ and $\Xi_1\psi^*(\cdot, 1)$ are optimal allocations for the stationary target problem with distribution $g(\cdot, 1)$ and exponential detection function. By the uniqueness of such solutions $\Xi_1\psi^* = \psi^*$. By a similar argument $\Xi_2\Xi_1\psi^* = \Xi_2\psi^* = \psi^*$ and, in fact, $\Xi_t\psi^* = \psi^*$ for $t = 1, \ldots, T$. Thus, for each $t = 1, \ldots, T$, $\psi^*(\cdot, t)$ maximizes the probability of detection for a stationary target with distribution $g_{\psi^*}(\cdot, t)$ and exponential detection function under the effort constraint $m(t)$. So ψ^* satisfies the basic necessary and sufficient condition given in the second section of Chapter I and is therefore T-optimal.

Constrained Markovian Motion

In this section we discuss a method which allows the user to specify constraining distributions for the target's location at times $t_1, t_2, \ldots, t_n \in \{1, \ldots, T\}$. Normally, the search planner specifies an initial distribution and a transition matrix to identify a Markov process which represents the target's motion. The initial distribution may be obtained from a long range detection with poor localization and the transition matrix may be developed from general knowledge of the target's behavior. However, the search planner may have information which leads him to believe that the target will have known distributions r_{t_i} for $i = 1, \ldots, n$ where $1 \le t_1 < \ldots < t_n \le T$. In this section we discuss a method for modifying a given transition function τ to obtain a Markov chain $\{\tilde{X}_t ; t = 0, \ldots, T\}$ such that $\tilde{X}_{t_i}$, the target's position at time t_i, has distribution r_{t_i} for $i = 1, \ldots, n$.

To begin, we consider the situation in which the target's distribution is specified at time $t = 1$ and $t = T$. (For this discussion we will assume that there is only one motion scenario and drop the scenario index from r, τ, and q.) We consider target path probabilities $q(\omega)$ which are computed by

$$q(\omega) = \rho(\omega_1)\tau_1(\omega_1, \omega_2)\tau_2(\omega_2, \omega_3) \ldots \tau_{T-1}(\omega_{T-1}, \omega_T)\rho'(\omega_T), \qquad \text{(II-5)}$$

where

$$\rho(j),\ \rho'(j) \ge 0 \qquad \text{for } j \in J.$$

In most cases, ρ and ρ' will not be probability distributions. If ρ is a probability distribution, then equation (I-5) may be interpreted as taking a Markovian motion

with prior distribution ρ and transition matrix τ and combining this according to Dempster's rule (see references [n] and [o]) with an independent estimate of the target position at time T which is represented by the probability distribution which is proportional to ρ'. The information in ρ' will, in general, change our estimate of where the target was at time zero. It will also change our estimate of how the target made its transitions. If, for example, ρ' is a more concentrated distribution than the distribution of a target which diffuses from ρ according to τ, then the target's transitions for times near T will have to have a bias toward the mean of ρ' to reconcentrate the target's density.

In order to compute the prior distribution and transition probabilities which result from this constrained model, we introduce the functions G and H where $G: \{1, \ldots, T\} \times J \times J \to [0, 1]$ is defined recursively by

$$G(1, j, j) = 1, G(1, i, j) = 0 \quad \text{if } i \neq j$$

$$G(t, i, j) = \sum_{k \in J} G(t-1, i, k)\, \tau_{t-1}(k, j) \quad \text{for } 2 \leq t \leq T$$

and $H: \{1, \ldots, T\} \times J \times J \to [0, 1]$ is defined recursively by

$$H(T, j, j) = 1, \; H(T, i, j) = 0 \quad \text{if } i \neq j$$

$$H(t, i, j) = \sum_{k \in J} \tau_t(i, k)\, H(t+1, k, j) \quad \text{for } 1 \leq t \leq T-1.$$

For a Markov chain governed by the transition function τ, $G(t, i, j)$ is the probability that the chain is in state j at time t given it was in state i at time 1. Similarly, $H(t, i, j)$ is the probability that the chain is in state j at time T given it was in state i at time t.

Let

$$r(i) = \sum_{j \in J} \rho(i)\, H(1, i, j)\, \rho'(j) \quad \text{for } i \in J. \tag{II-10}$$

From the definition of H and the path probabilities q, one can see that $r(i)$ is the sum of the probabilities of all paths which start at i, so $r(i)$ is the probability that the target starts in cell i. Likewise, the probability that the target ends in cell j is given by

$$r'(j) = \sum_{i \in J} \rho(i)\, H(1, i, j)\, \rho'(j). \tag{II-11}$$

The imposition of the weights ρ and ρ' on the sample path probabilities causes the transition probabilities of the Markov chain to be modified. Let $\tilde{\tau}$ indicate the modified transition function and $\{\tilde{X}_t \,;\, t = 0, \ldots, T\}$ the stochastic process with path probabilities given by q in (II-5). Then

$$\tilde{\tau}_t(i, j) = \Pr\{\tilde{X}_t = i \text{ and } \tilde{X}_{t+1} = j\} / \Pr\{\tilde{X}_t = i\}, \tag{II-12}$$

where

$$\Pr\{\tilde{X}_t = i \text{ and } \tilde{X}_{t+1} = j\} = \sum_{\{\omega : \omega_t = i \text{ and } \omega_{t+1} = j\}} q(\omega)$$

$$= \sum_{k \in J} \sum_{k' \in J} \rho(k)\, G(t, k, i)\, \tau_t(i, j)\, H(t+1, j, k')\, \rho'(k'),$$

and

$$\Pr\{X_t = i\} = \sum_{k \in J} \sum_{k' \in J} \rho(k)\, G(t, k, i)\, H(t, i, k')\, \rho'(k').$$

Thus

$$\tilde{\tau}_t(i,j) = \tau_t(i,j) \left[\sum_{k' \in J} H(t+1,j,k')\, \rho'(k')\right] \Big/ \left[\sum_{k' \in J} H(t,i,k')\, \rho'(k')\right]. \qquad \text{(II-}$$

From equations (II-10) and (II-13) one can calculate that

$$\begin{aligned} \Pr\{\omega\} &= r(\omega_1) \prod_{t=1}^{T-1} \tilde{\tau}_t(\omega_t, \omega_{t+1}) \\ &= \rho(\omega_1) \prod_{t=1}^{T-1} \tilde{\tau}_t(\omega_t, \omega_{t+1})\, \rho'(\omega_T) \\ &= q(\omega). \end{aligned}$$

Thus the transition function $\tilde{\tau}$ along with the initial distribution r produces the path probabilities q defined in (II-5).

The search planner's estimate of the target's local motion characteristics is normally in terms of τ. When the searcher's estimate of the target's prior distribution comes from a contact with an associated uncertainty, which is based on the characteristics of the detector and the detectability of the target, but not on the subsequent motion of the target, then this is an estimate of ρ. On the other hand, when the searcher's estimate of the target's prior distribution comes from historical information which accounts for the target's subsequent motion, then this is an estimate of r. Thus the searcher may reasonably start with either r or ρ. Likewise, he may start with either r' or ρ'.

The program described in reference [a] handles four forms of input; ρ, τ, ρ'; ρ, τ, r'; r, τ, ρ'; and r, τ, r'. Computationally, the form ρ, τ, ρ' is easiest to work with so we convert the other inputs to this form. When ρ, τ, and r' are entered, we use equations (II-8) and (II-9) to compute H and then solve equation (II-11) for ρ'.

The solution is just

$$\rho'(j) = r'(j) / \sum_{k \in J} \rho(k)\, H(1,k,j)$$

for all $j \in J$. The denominator of this expression can vanish only when it is physically impossible for a target which starts in a cell k with $\rho(k) > 0$ and makes only transitions which τ gives nonzero probability to get to cell j at time T. If $r'(j)$ also vanishes, we can set $\rho'(j) = 0$. On the other hand, if $r'(j) \neq 0$, then the inputs describe an impossible situation and no solution is possible. Likewise when r, τ, and ρ' are entered, we compute H and then solve equation (II-10) for ρ. Again there is no solution only in a physically impossible situation.

The final input possibility; r, τ, and r'; presents greater difficulties. We compute H as above and then solve the quadratic equations (II-10) and (II-11) for ρ and ρ'. This is accomplished by the following iterative algorithm:

(1) Let ϵ be a small positive number.

(2) For all $j \in J$ set $\rho_0(j) = r(j)$.

(3) For all $j \in J$ set $\rho'_0(j) = r'(j)$.

(4) Set $l = 0$.

(5) For all $j \in J$ set $\rho_{l+1}(j) = r(j) / \left[\sum_{k \in J} H(1,j,k)\, \rho'_l(k)\right]$.

(6) For all $j \in J$ set $\rho'_{l+1}(j) = r'(j) / \left[\sum_{k \in J} \rho_{l+1}(k)\, H(1,k,j)\right]$.

(7) If $|\rho_{l+1}(j) - \rho_l(j)| < \epsilon$ and $|\rho'_{l+1}(k) - \rho'_l(k)| < \epsilon$ for all $j, k \in J$, then stop: The answer is $\rho = \rho_{l+1}$ and $\rho' = \rho'_{l+1}$.

(8) Otherwise, set $l = l+1$.

(9) Go back to step (5).

As above, a zero should be supplied for the result of an indicated division of zero by zero in steps (5)-(6). While no convergence results are available for this algorithm, it has proven effective in examples such as those given in the second section.

More generally, we may have information about the target's distribution at a sequence of time intervals $1 \leq t_1 < t_2 < \ldots < t_n \leq T$. For this case we consider a more general definition of q as follows:

$$q(\omega) = \rho_1(\omega_1) \prod_{t=1}^{T-1} [\tau_t(\omega_t, \omega_{t+1}) \rho_{t+1}(\omega_{t+1})] \quad \text{for } \omega \in \Omega.$$

For $t \neq t_1, \ldots, t_l$, we take $\rho_t(j) = 1$ for $j \in J$. For t_n, $n = 1, \ldots, l$, ρ_{t_n} is a non-negative function on J which is related to our information about the target's distribution at the times t_n, $n = 1, \ldots, l$. That is the user may specify ρ_{t_n} for $t_n = 1, \ldots, l$ or he may specify r_{t_n} the target's distribution at time t_n and solve for ρ_{t_n} as above for $n = 1, \ldots, l$.

Once we have solved for or obtained the ρ_t for $t = 1, \ldots, T$, we can calculate $q(\omega)$ as though it were obtained from a Markov chain with initial distribution ρ_1 and transition function τ' defined by

$$\tau'_t(i, j) = \tau_t(i, j)\, \rho_{t+1}(j) \quad \text{for } t = 1, \ldots, T-1 \text{ and } i, j \in J,$$

even though ρ_1 may not be a probability distribution and τ' may be a defective transition function. This causes no problem because the probabilities computed in this fashion are all proportional to the correct probabilities. In particular the target location distribution at any t is proportional to the one obtained from this Markov

chain. The algorithm will give the same allocations of effort for any two distributions which differ by a constant factor.

Our final observation concerning the motion model is that it can be used to create multiple scenarios. Each state can, for example, consist of a cell and a scenario index. The target whose state consists of cell j and scenario n moves by applying a transition matrix associated with scenario n to j to obtain its new cell. It always keeps the same scenario. The initial scenario weights are incorporated in the amount of target mass which the prior distribution on the states places in states with each scenario index.

CHAPTER III

RECTANGULAR SEARCH PLANS

In this chapter we discuss rectangular search plans, which model sonobuoy fields, to search for a target moving in discrete space and in discrete time with a fixed time limit, T. The computer algorithm, which generates the rectangular search plans, is described and the effectiveness of these rectangular plans is measured by comparison (via computed examples) with T-optimal plans. In the last section of this chapter we summarize outstanding problems associated with rectangular barrier search. All of the examples in this chapter are taken from references [c] and [d].

The Search Problem

The search space, search time, and target motion assumptions are as in Chapter II with the exception that at each time interval the searcher must spread his limited amount of effort uniformly over a rectangular region. The probability of detecting the target during the t^{th} time interval if rectangle R is searched is given by

$$[1 - \exp(-m(t)/A(R))] \cdot p(R)$$

where

$m(t)$ = total effort available at time t

$A(R)$ = area of the rectangle R

and

$$p(R) = \text{probability that the target is located in } R.$$

The number of time intervals, T, is specified at the outset and the searcher attempts to choose a rectangle to search at each time interval to maximize the probability that he will detect the target within the specified time limit. We refer to an allocation of effort which accomplishes this objective as an optimal rectangular plan to distinguish it from the T-optimal plans found in Chapter II. When the number of time intervals is one, we refer to the problem of choosing an optimal rectangle as a rectangular stationary target problem.

Rectangular Barrier Algorithm

In this section we describe the computer algorithm which computes the rectangular search plans given in the examples in the next section. Although we do not claim that our rectangular plans are optimal rectangular plans, experimental comparison with T-optimal plans shows that our plans cannot be far from rectangular optimality since in fact these plans are not far from T-optimality.

The motivation for the approximations made by our algorithms may be found in reference [p], where the problem of allocating a fixed amount of effort uniformly over a rectangle to find a stationary target, whose location distribution is bivariate normal, is considered. Suppose the bivariate normal distribution has standard deviations σ_1 and σ_2 along the major and minor axes, respectively. Reference [p] then considers only rectangles centered at the mean of the distribution, oriented along the major axis of the distribution, and whose length and width are proportional to σ_1 and σ_2, respectively. The resulting one variable problem can be solved by

standard calculus. Comparison of the detection probabilities of the search plan thus produced to the totally optimal detection probability shows that the approximations involved are extremely good (see reference [q]).

With the above motivation in mind, we can now state our rectangular barrier algorithm. Our algorithm follows the description of the basic algorithm as given in the third section of Chapter I, except that it attempts to solve a rectangular stationary target problem at each time step instead of an ordinary stationary target problem. This being the case, we need only describe the workings of our algorithm at a fixed time step on the rectangular stationary target problem which results from motion as well as failure to detect at all other time periods.

The rectangular stationary target problem is a problem in maximizing a function of five variables given by the length, width, orientation, and center of the rectangle. In general, this function need not be convex, so a straightforward attempt to find the maximum by computer algorithms would be difficult. Instead, we try to take advantage of the intuition exhibited in reference [p] to make some good choices for some of the variables, thereby reducing the number of variables in the optimization. Thus, at each time in the search we proceed as follows. Check whether the distribution is unimodal or not. If it is unimodal we consider a single class of rectangles, while if the distribution is multimodal we consider three classes of rectangles. In the unimodal case we restrict ourselves to the class of rectangles that are centered at the mean of the distribution and oriented along the major axis of the distribution. The major axis of the distribution is obtained by computing the covariance matrix of the distribution and finding the angle of rotation that diagonalizes the matrix. This corresponds to the rotation of coordinates which transforms a normal distribution

with that covariance matrix into the product of two independent one-dimensional normal distributions along the rotated axes. The axis having the distribution with the largest variance is the major axis. We then optimize over the two remaining variables--length and width.

In the multimodal case, we consider the above class of rectangles and two additional classes. To determine the additional classes we compute the two best modes of the distribution. We then mimic . was done in the unimodal case for the distribution near each of these modes. To do this we assign cells to be in the "local distribution" of one of these modes as follows. Proceed radially from the mode until the probabilities begin to increase. Cut off the "local distribution" at this point. The resulting grid of cells is what we call the "local distribution" of that mode. We now renormalize these "local distributions" and compute the mean and covariance matrix of these two "local distributions." This leads to two further classes of rectangles, and proceeding in the manner described above, we find the best rectangle in each of these classes. We now have three "best" rectangles, one based on each of the two best modes and one centered at the mean of the distribution. Finally, we take the best rectangle from among these three best rectangles.

Examples

In this section we discuss three examples of rectangular search plans.

Example 1: Markovian fan. In our first example, the target motion is identical to that in Example 1 of Chapter II.

Table 1 below shows the target density, updated by search effort, and our rectangles for 8 hours. The target moves southward and diffuses from its

TABLE III-1

PROBABILITY MAPS AND RECTANGULAR SEARCH PLAN FOR EXAMPLE 1 MARKOVIAN FAN

Notes: (1) Entries represent thousandths of a target.
(2) The rectangle for Hour 1 is the square which surrounds the entire distribution.
(3) Cells are 3 mi by 3 mi.

PROBABILITY MAP AND RECTANGLE FOR HOUR 1

N ↑

Hour 1

23	34	[illegible]	34	23
34	[illegible]	[illegible]	[illegible]	34
[illegible]	[illegible]	63	[illegible]	[illegible]
[illegible]	45	[illegible]	[illegible]	34
[illegible]	34	[illegible]	34	23

Vertical axis: 2, 1, [illegible], −1, −2, −3

Horizontal axis: 2, 1, [illegible], −1, −2, −3

TIME 1 PROBABILITY OF DETECTION = .329

PROBABILITY MAP AND RECTANGLE FOR HOUR 2

	[illegible]	[illegible]	[illegible]	11	[illegible]	[illegible]	[illegible]	[illegible]
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1
1	[illegible]	[illegible]	44	[illegible]	[illegible]	[illegible]	[illegible]	1
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13	[illegible]
1	11	[illegible]	[illegible]	[illegible]	41	[illegible]	11	1
1	7	[illegible]	23	[illegible]	23	15	7	1
[illegible]	1	[illegible]	[illegible]	5	[illegible]	[illegible]	1	[illegible]

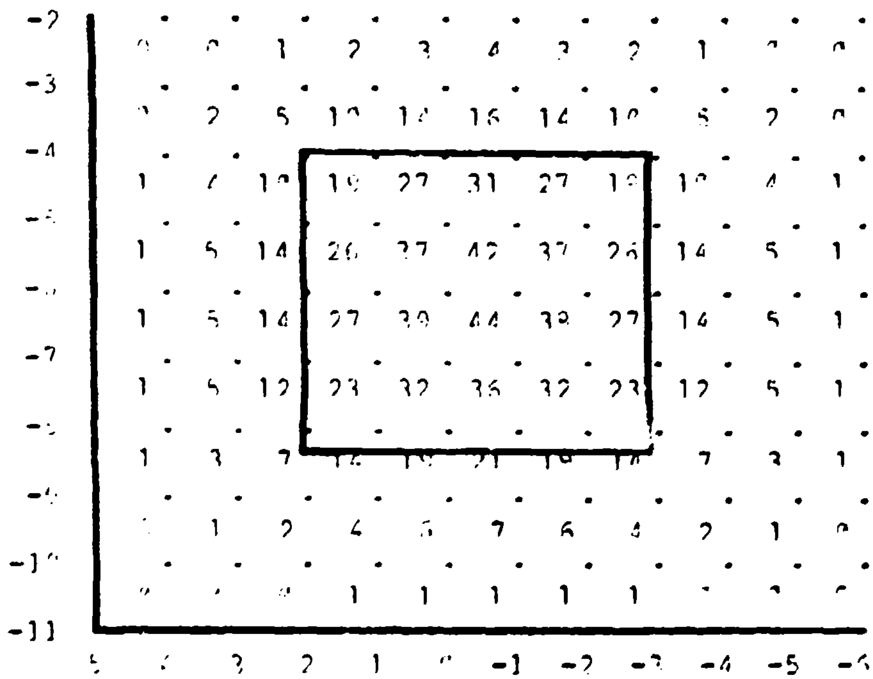
Vertical axis: −1, −2, −3, −4, [illegible], [illegible], −7

Horizontal axis: [illegible], [illegible], [illegible], 1, [illegible], −1, −2, [illegible], [illegible], [illegible]

TIME 2 PROBABILITY OF DETECTION = .461 (Cumulative)

TABLE III 1 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 3



TIME 3 PROBABILITY OF DETECTION = .599 (Cumulative)

TABLE III-1 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 4

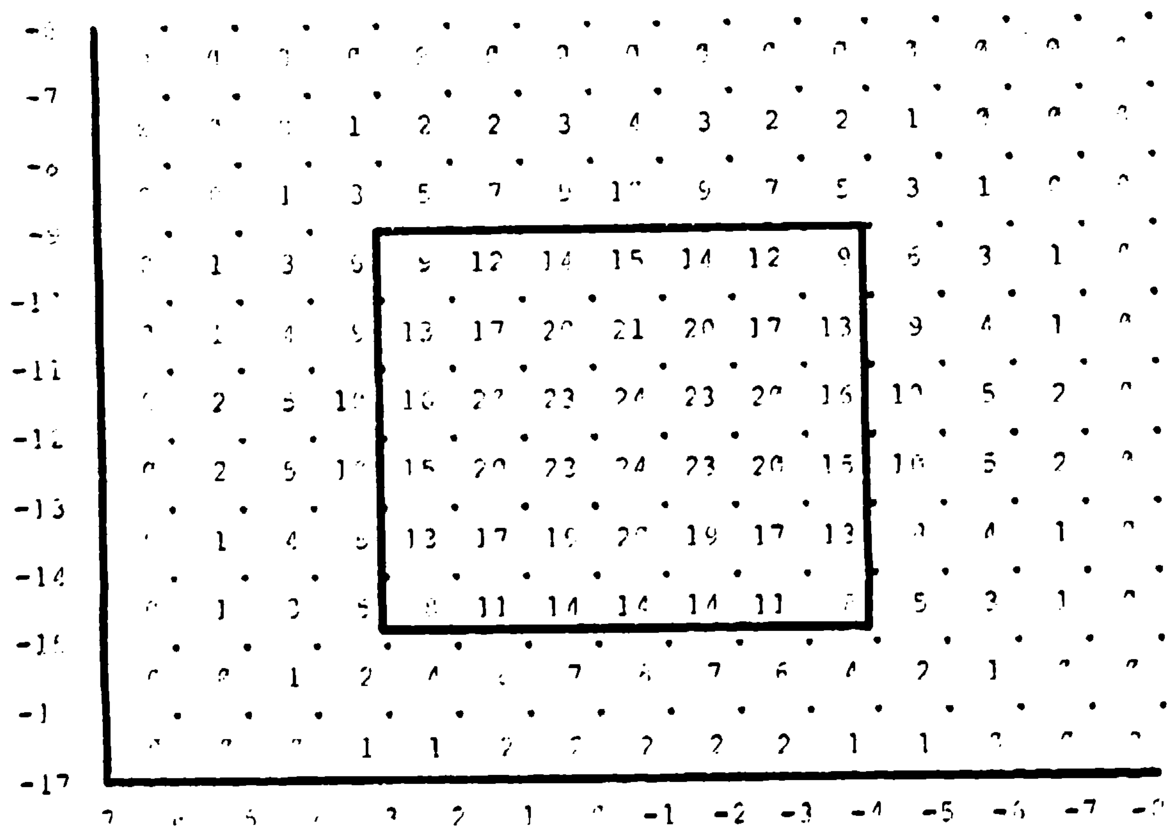
-4	0	0	0	0	1	1	1	1	1	0	0	0	0
-5	0	0	1	3	5	7	9	7	5	3	1	0	0
-6	0	1	4	8	12	16	18	15	12	8	4	1	0
-7	0	2	7	12	18	23	25	22	18	12	7	2	0
-8	1	3	8	15	22	28	30	29	22	15	8	3	1
-9	1	3	9	15	23	28	31	28	23	15	9	3	1
-10	1	3	7	13	19	24	27	24	19	13	7	3	1
-11	0	2	4	8	13	17	19	17	13	8	4	2	0
-12	0	1	2	3	5	7	8	7	5	3	2	1	0
-13	0	0	0	1	1	2	2	2	1	1	0	0	0
-14													

7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7

TIME 4 PROBABILITY OF DETECTION = .670 (Cumulative)

TABLE III-1 (continued)

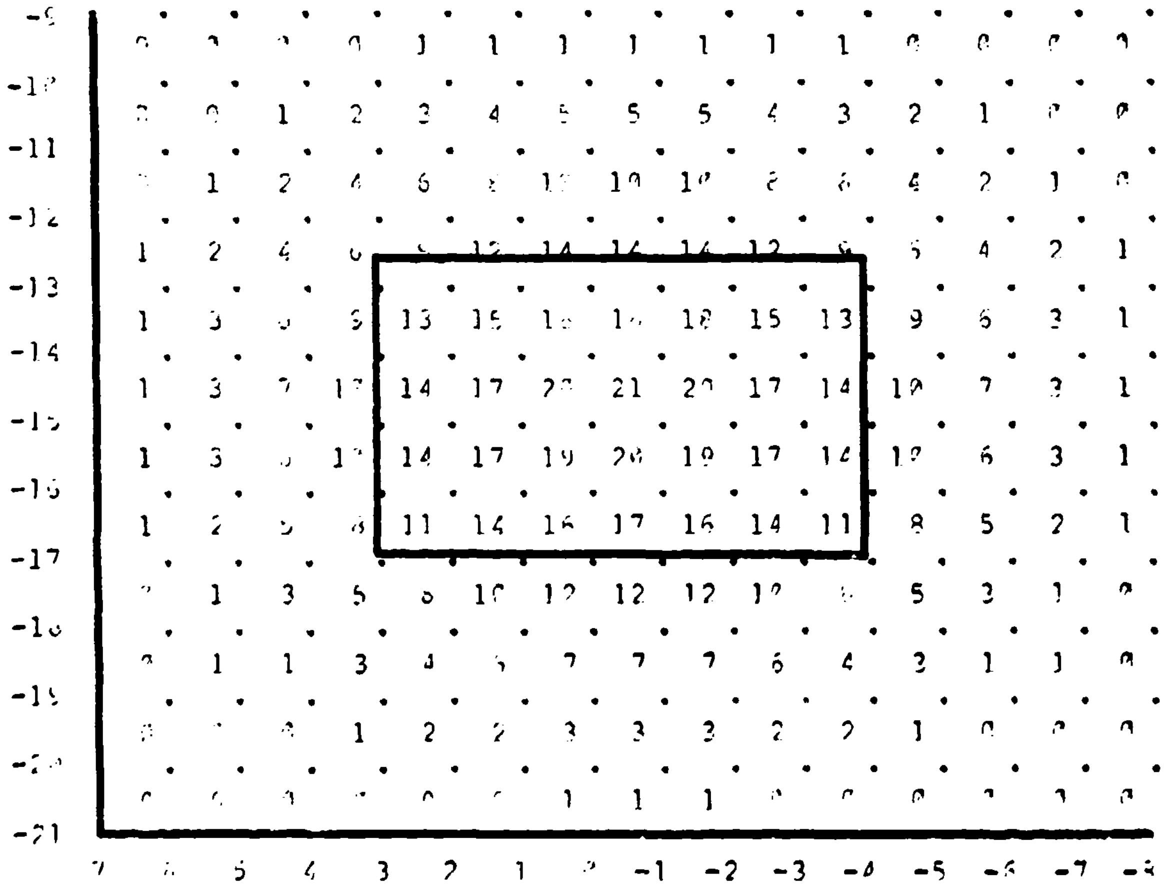
PROBABILITY MAP AND RECTANGLE FOR HOUR 5



TIME 5 PROBABILITY OF DETECTION = .718 (Cumulative)

TABLE III-1 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 6



TIME 6 PROBABILITY OF DETECTION = .764 (Cumulative)

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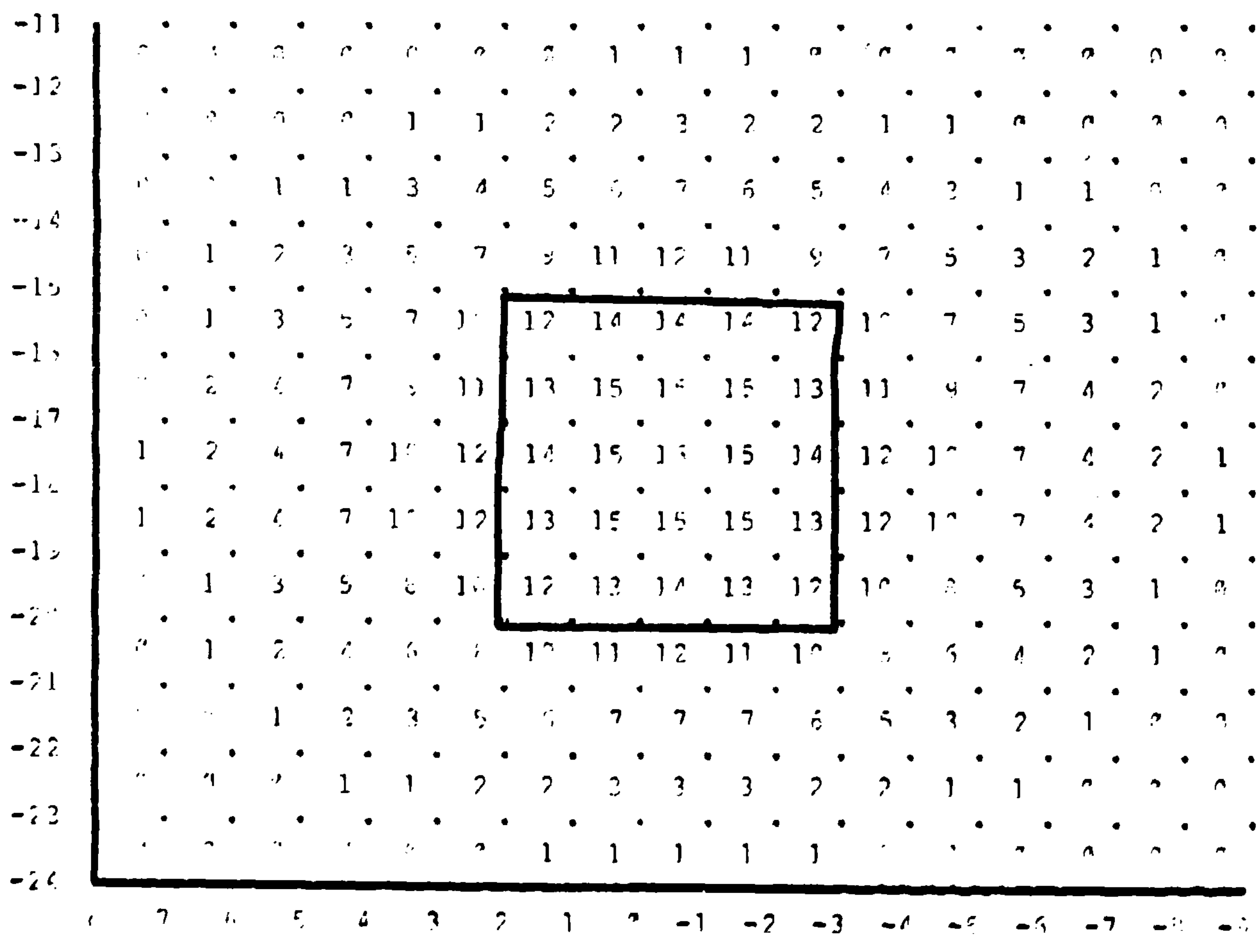
TABLE III-1 (continued)

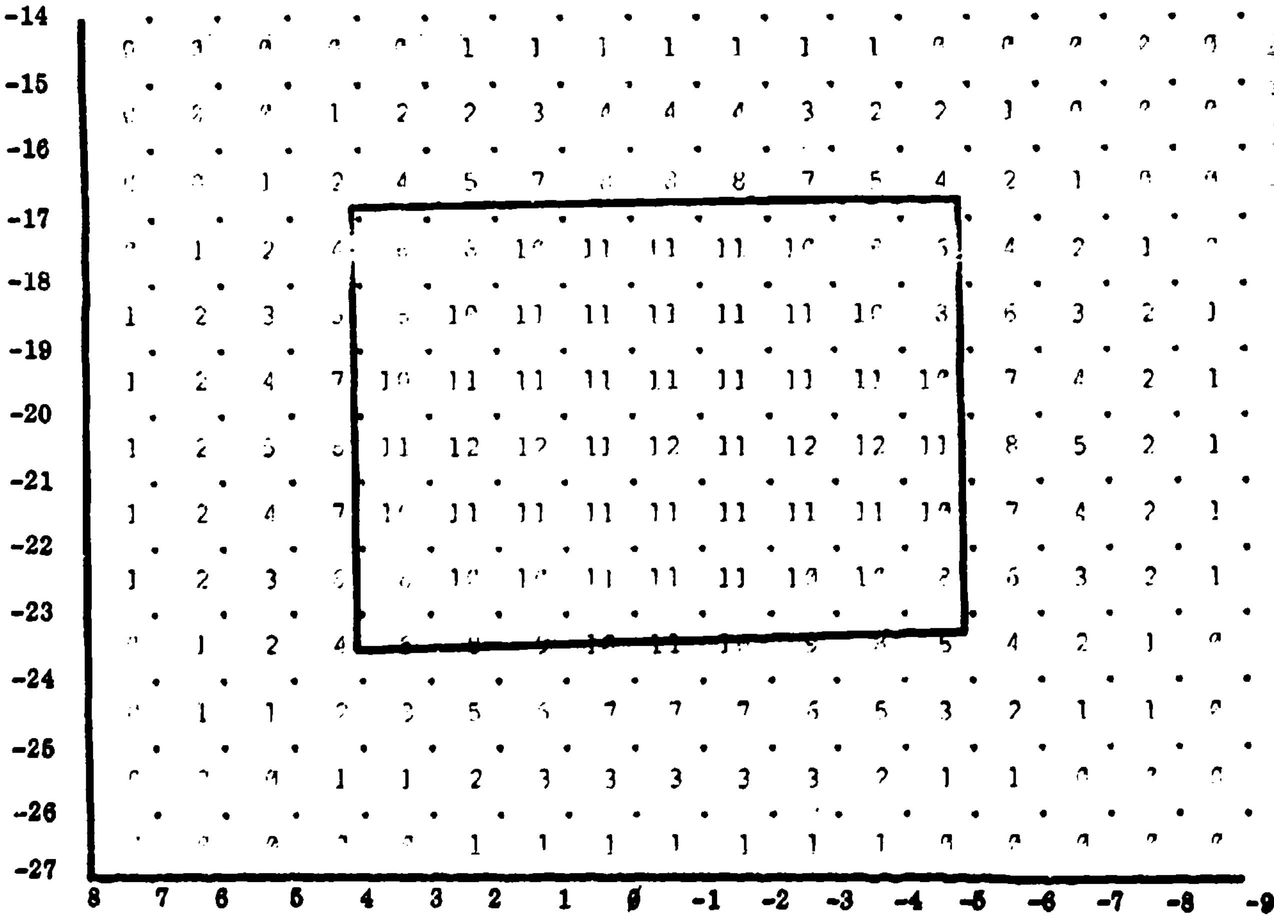
PROBABILITY MAP AND RECTANGLE FOR HOUR 7



TIME 7 PROBABILITY OF DETECTION = .781 (Cumulative)

TABLE III-1 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 8



TIME 8 PROBABILITY OF DETECTION = .802 (Cumulative)

original distribution. The searcher has 90 units of search effort available each hour which he applies uniformly over the rectangles shown.

As can be seen from Table III-1, the rectangles chosen for all 8 time periods are intuitively very reasonable. Table III-2 below gives the detection probabilities for each time interval for our rectangular plan and optimal plan for eight hours. As Table III-2 shows, the rectangular plan has detection probability only .017 less than the detection probability of the totally optimal plan.

<u>Example II: Constrained Markovian fan.</u> We now modify the previous example by assuming that historical information indicates the target's distribution at time 8 as well as at time 1. In particular, we assume that at hour 8 the target's prior distribution is identical to the initial distribution but translated 60 miles south. The motion in this example is identical to that in Example 2 of Chapter II. Note that the distribution which appears in Table III-3 below is not circular normal at time 8 because search effort has been applied during hours 1 through 7 inside the indicated rectangles.

For each of hours 1 through 8, Table III-3 shows the target distribution conditioned on failure to detect prior to that time along with the search rectangle for that hour. The target initially diffuses, but by hour 5 it must start to reconcentrate its density to attempt to meet the constraint at hour 8.

Because of the constraint at hour 8, the posterior distributions are more concentrated so that guessing the rectangles by inspection might appear to be easy. However, choosing by inspection can lead to mistakes. For instance, one might guess that at hour 2, the 5 x 5 rectangle with vertices (2, -1), (2, -6), (-2, -6), and (-2, -1) is a better rectangle than the one chosen by the algorithm. However, one

TABLE III-2

DETECTION PROBABILITIES FOR EXAMPLE 1

	Probability of Detection for	
Time	Rectangular Plan	Optimal Plan for 8 Hours
1	.329	.311
2	.481	.515
3	.599	.624
4	.670	.692
5	.718	.739
6	.754	.773
7	.781	.798
8	.802	.818

TABLE III-3

PROBABILITY MAPS AND RECTANGULAR SEARCH PLAN FOR EXAMPLE 2: CONSTRAINED MARKOVIAN FAN

Notes: (1) Entries represent thousandths of a target.
(2) The rectangle for hour 1 is the square which surrounds the entire distribution.
(3) Cells are 3 mi by 3 mi

PROBABILITY MAP AND RECTANGLE FOR HOUR 1

N ↑

2					
	23	34	38	34	23
1					
	34	49	56	49	34
0					
	38	56	63	56	38
-1					
	34	49	56	49	34
-2					
	23	34	38	34	23
-3					
	2	1	0	-1	-2 -3

TIME 1 PROBABILITY OF DETECTION = .329

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 2

0	1	3	7	7	7	3	1	0
0	6	17	31	36	31	17	6	0
1	9	27	50	58	50	27	9	1
1	10	30	55	64	55	30	10	1
1	9	25	47	54	47	25	9	1
0	4	12	23	26	23	12	4	0
0	0	2	3	3	3	2	0	0

0 -1 -2 -3 -4 -5 -6

4 3 2 1 0 -1 -2 -3 -4 -5

TIME 2 PROBABILITY OF DETECTION = .539 (Cumulative)

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 3

-2										
	0	0	1	1	2	1	1	3	1	
-3										
	1	3	7	10	11	10	7	3	1	
-4										
	2	9	19	20	20	20	19	9	2	
-5										
	4	15	31	44	45	44	31	15	4	
-6										
	4	16	33	47	48	47	33	16	4	
-7										
	3	12	26	38	40	38	26	12	3	
-8										
	1	5	12	19	21	19	12	5	1	
-9										
	0	1	2	4	4	4	2	1	0	
-10										
	4	3	2	1	0	-1	-2	-3	-4	-5

TIME 3 PROBABILITY OF DETECTION = .660 (Cumulative)

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 4

0	0	1	2	3	3	3	2	1	0	0
0	1	5	10	15	17	15	10	5	1	0
1	4	11	23	32	36	32	23	11	4	1
1	5	15	29	38	42	38	29	15	5	1
1	5	15	29	37	41	37	29	15	5	1
1	4	11	23	32	36	32	23	11	4	1
0	2	5	12	18	21	18	12	5	2	0
0	0	1	2	4	5	4	2	1	0	0

Vertical axis: -5 -6 -7 -8 -9 -10 -11 -12 -13

Horizontal axis: 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6

TIME 4 PROBABILITY OF DETECTION = .744 (Cumulative)

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 5

-8	0	0	1	3	5	5	5	3	1	0	0
-9	0	2	6	13	19	22	19	13	6	2	0
-10	1	4	12	23	31	35	31	23	12	4	1
-11	1	6	16	28	35	38	35	28	16	6	1
-12	1	6	15	27	34	37	34	27	15	6	1
-13	1	4	11	22	31	35	31	22	11	4	1
-14	0	1	5	11	18	21	18	11	5	1	0
-15	0	0	1	2	3	4	3	2	1	0	0
-16											

5 4 3 2 1 0 -1 -2 -3 -4 -5 -6

TIME 5 PROBABILITY OF DETECTION = .800 (Cumulative)

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 6

-11	0	0	2	3	5	6	5	3	2	0	0
-12	0	2	8	15	20	20	20	15	8	2	0
-13	1	4	14	27	33	32	33	27	14	4	1
-14	1	5	17	31	37	35	37	31	17	5	1
-15	1	5	16	30	36	34	36	30	16	5	1
-16	0	3	12	23	30	30	30	23	12	3	0
-17	0	1	4	10	14	15	14	10	4	1	0
-18	0	0	0	1	2	2	2	1	0	0	0
-19											

5 4 3 2 1 0 -1 -2 -3 -4 -5 -6

TIME 6 PROBABILITY OF DETECTION = .846 (Cumulative)

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 7

-14 / -15	0	1	4	6	6	6	4	1	0
-15 / -16	1	9	20	28	30	28	20	9	1
-16 / -17	2	14	29	36	36	36	29	14	2
-17 / -18	2	16	30	36	37	36	30	16	2
-18 / -19	2	15	29	36	36	36	29	15	2
-19 / -20	1	11	23	32	34	32	23	11	1
-20 / -21	0	3	7	12	13	12	7	3	0

4 3 2 1 0 -1 -2 -3 -4 -5

TIME 7 PROBABILITY OF DETECTION = .884 (Cumulative)

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 8

-18						
	39	43	45	43	39	
-19						
	40	40	39	40	40	
-20						
	41	39	37	39	41	
-21						
	39	38	36	38	39	
-22						
	37	42	44	42	37	
-23						
	2	1	0	-1	-2	-3

TIME 8 PROBABILITY OF DETECTION = .922 (Cumulative)

can check that the probability of detection using this rectangle is .52194, which is not as good as the rectangle chosen by the computer.

Table III-4 compares the detection probabilities for the optimal and rectangular plans for each of the eight hours. As can be seen, the rectangular plan never is significantly behind either the myopic or optimal plans. The rectangular plan has a final detection probability only .010 less than the optimal plan for eight hours.

Example 3: Radial flee. In our last example the target starts from the same prior distribution used in the two previous examples. Its speed is chosen uniformly from the interval [6, 12]. To choose its course, it first chooses one of eight base headings: 22.5°, 67.5°, 112.5°, 157.5°, 202.5°, 247.5°, 292.5°, or 337.5°. Having chosen one of these base headings it keeps it for the entire five time units of the search. Once a base heading is chosen the target's course at each time interval is chosen uniformly from whichever of the following intervals contains the base heading: $[0^{\circ}, 45^{\circ}]$, $[45^{\circ}, 90^{\circ}]$, $[90^{\circ}, 135^{\circ}]$, $[135^{\circ}, 180^{\circ}]$, $[180^{\circ}, 225^{\circ}]$, $[225^{\circ}, 270^{\circ}]$, $[270^{\circ}, 315^{\circ}]$, or $[315^{\circ}, 360^{\circ}]$. This example approximates the classical radial flee target motion discussed in reference [1].

As can be seen in Table III-5 the target diffuses very rapidly and for hours 3, 4, and 5 the target's distribution is clearly multimodal. The rectangles chosen during these times are clearly those from the class of rectangles corresponding to one of the two best modes. Table III-6 compares the rectangular plan and the optimal plan for each of the 5 time intervals.

Outstanding Problems

In this section we summarize outstanding problems associated with the rectangular barrier problem.

TABLE III-4

Probability of Detection for

Time	Rectangular Plan	8-Interval Optimal Plan
1	.329	.311
2	.539	.540
3	.660	.672
4	.744	.757
5	.800	.816
6	.846	.862
7	.884	.898
8	.922	.932

TABLE III-5

RECTANGULAR PLAN FOR RADIAL FLEE

PROBABILITY MAP AND RECTANGLE FOR HOUR 1

Note: **Entries represent thousandths of a target.**

23	34	38	34	23
34	49	56	49	34
38	56	63	56	38
34	49	56	49	34
23	34	38	34	23

Vertical axis: 2 1 0 -1 -2 -3

Horizontal axis: 2 1 0 -1 -2 -3

PROBABILITY OF DETECTION = .329

TABLE III-6 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 2

6													
	0	0	0	0	1	1	1	1	1	0	0	0	0
5													
	0	0	1	2	3	4	5	4	3	2	1	0	0
4													
	0	1	2	5	8	10	11	10	8	5	2	1	0
3													
	0	2	5	8	11	13	14	13	11	8	5	2	0
2													
	1	3	8	11	14	15	15	15	14	11	8	3	1
1													
	1	4	10	13	15	14	14	14	15	13	10	4	1
0													
	1	5	11	14	15	14	13	14	15	14	11	5	1
-1													
	1	4	10	13	15	14	14	14	15	13	10	4	1
-2													
	1	3	8	11	14	15	15	15	14	11	8	3	1
-3													
	0	2	5	8	11	13	14	13	11	8	5	2	0
-4													
	0	1	2	5	8	10	11	10	8	5	2	1	0
-5													
	0	0	1	2	3	4	5	4	3	2	1	0	0
-6													
	0	0	0	0	1	1	1	1	1	0	0	0	0
-7													

6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7

PROBABILITY OF DETECTION = .411 (Cumulative)

TABLE III-3 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 3

Ø	Ø	Ø	Ø	Ø	Ø	1	1	1	1	1	1	1	Ø	Ø	Ø	Ø	Ø	Ø
Ø	Ø	Ø	Ø	1	1	2	2	3	3	3	2	2	1	1	Ø	Ø	Ø	Ø
Ø	Ø	Ø	1	2	3	5	5	5	5	5	5	5	3	2	1	Ø	Ø	Ø
Ø	Ø	1	1	3	5	6	7	7	7	7	7	6	5	3	1	1	Ø	Ø
Ø	1	2	3	4	6	6	6	6	6	6	6	6	6	4	3	2	1	Ø
Ø	1	3	5	5	6	6	5	4	4	4	5	6	6	5	5	3	1	Ø
1	2	4	6	6	6	4	3	2	2	2	3	4	6	6	6	4	2	1
1	2	5	6	6	5	3	1	1	1	1	1	3	5	6	6	5	2	1
1	3	5	6	5	4	2	1	Ø	Ø	Ø	1	2	4	5	6	5	3	1
1	3	5	6	5	4	2	1	Ø	Ø	Ø	1	2	4	5	6	5	3	1
1	3	5	6	5	4	2	1	Ø	Ø	Ø	1	2	4	5	6	5	3	1
1	2	5	6	6	5	3	1	1	1	1	1	3	5	6	6	5	2	1
1	2	4	6	6	6	4	3	2	2	2	3	4	6	6	6	4	2	1
Ø	1	3	5	5	6	6	5	4	4	4	5	6	6	5	5	3	1	Ø
Ø	1	2	3	4	6	6	6	6	6	6	6	6	6	4	3	2	1	Ø
Ø	Ø	1	1	3	5	6	7	7	7	7	7	6	5	3	1	1	Ø	Ø
Ø	Ø	Ø	1	2	3	5	5	5	5	5	5	5	3	2	1	Ø	Ø	Ø
Ø	Ø	Ø	Ø	1	1	2	2	3	3	3	2	2	1	1	Ø	Ø	Ø	Ø
Ø	Ø	Ø	Ø	Ø	Ø	1	1	1	1	1	1	1	Ø	Ø	Ø	Ø	Ø	Ø

Vertical axis: 8 7 6 5 4 3 2 1 Ø -1 -2 -3 -4 -5 -6 -7 -8 -9 -1Ø

Horizontal axis: 9 8 7 6 5 4 3 2 1 Ø -1 -2 -3 -4 -5 -6 -7 -8 -9 -1Ø

PROBABILITY OF DETECTION = .434 (Cumulative)

TABLE III-5 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 4

12	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	[illegible]
11	0	0	0	0	0	0	1	1	2	2	2	1	1	1	2	2	2	1	1	0	0	0	0	0	[illegible]
10	0	0	2	0	0	1	1	3	3	4	3	2	2	2	3	4	3	3	1	1	0	0	0	0	[illegible]
9	0	0	0	0	0	1	2	4	4	4	4	3	2	3	4	4	4	4	2	1	0	0	0	0	[illegible]
8	0	0	0	0	1	1	2	4	4	4	3	2	2	2	3	4	4	4	2	1	1	0	0	0	[illegible]
7	0	0	1	1	1	2	2	3	3	3	3	2	1	2	3	3	3	3	2	2	1	1	1	0	0
6	0	1	1	2	3	3	2	2	2	2	2	1	1	1	2	2	2	2	2	3	3	2	1	1	[illegible]
5	0	1	3	4	4	4	3	2	1	1	1	1	1	1	1	1	1	2	3	4	4	4	3	1	[illegible]
4	0	2	4	5	5	4	3	1	1	0	0	0	0	0	0	0	1	1	3	4	5	5	4	2	[illegible]
3	1	2	4	5	5	4	3	1	0	0	0	0	0	0	0	0	0	1	3	4	5	5	4	2	1
2	1	2	3	4	4	4	2	1	0	0	0	0	0	0	0	0	0	1	2	4	4	4	3	2	1
1	0	1	2	3	3	2	2	1	0	0	0	0	0	0	0	0	0	1	2	2	3	3	2	1	0
0	0	1	2	3	2	2	1	1	0	0	0	0	0	0	0	0	0	1	1	2	2	3	2	1	[illegible]
-1	0	1	2	3	3	2	2	1	0	0	0	0	0	0	0	0	0	1	2	2	3	3	2	1	[illegible]
-2	1	2	3	4	4	4	2	1	0	0	0	[illegible]	0	0	0	0	0	1	2	4	4	4	3	2	[illegible]
-3	1	2	4	5	5	4	3	1	0	0	0	0	0	0	0	0	0	1	3	4	5	5	4	2	[illegible]
-4	0	2	4	5	5	4	3	1	1	0	0	0	0	0	0	0	1	1	3	4	5	5	4	2	[illegible]
-5	0	1	3	4	4	4	3	2	1	1	1	1	1	1	1	1	1	2	3	4	4	4	3	1	[illegible]
-6	0	1	1	2	3	3	3	3	3	3	2	2	1	2	2	3	3	3	3	3	3	2	1	1	[illegible]
-7	0	0	1	1	1	2	3	4	4	5	4	3	2	3	4	5	4	4	3	2	1	1	1	0	0
-8	0	0	0	0	1	1	3	4	6	6	5	3	3	3	5	6	6	4	3	1	1	0	0	0	0
-9	0	0	0	0	0	1	2	4	5	6	5	3	3	3	5	6	5	4	2	1	0	0	0	0	[illegible]
-10	0	0	0	0	0	1	1	3	4	4	3	3	2	3	3	4	4	3	1	1	0	0	0	0	[illegible]
-11	0	0	0	0	0	0	1	1	2	2	2	1	1	1	2	2	2	1	1	0	0	0	0	0	[illegible]
-12	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	[illegible]
-13																									
12	11	10	9	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-1

PROBABILITY OF DETECTION = .454 (Cumulative)

TABLE III-5 (continued)

PROBABILITY MAP AND RECTANGLE FOR HOUR 5

0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	2	3	3	2	2	1	1	1	2	2	3	3	2	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	2	3	4	4	3	2	1	1	1	2	3	4	4	3	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	2	3	4	4	3	2	1	1	1	2	3	4	4	3	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	2	3	3	3	3	2	1	1	1	2	3	3	3	3	2	1	0	0	0	0	0	0
0	0	0	0	0	1	1	1	2	2	2	2	2	1	1	0	1	1	2	2	2	2	2	1	1	1	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	2	3	2	2	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	2	2	3	2	1	1
0	1	2	3	4	4	3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	3	4	4	3	2	1
0	1	3	4	5	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	4	3	1
1	2	3	4	5	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	4	3	2
0	1	3	4	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	4	3	1
0	1	2	2	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	2	3	2	2	1
0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0
0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	1	2	2	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	2	3	2	2	1
0	1	3	4	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	4	3	1
1	2	3	4	5	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	4	3	2
0	1	3	4	5	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	4	3	1
0	1	2	3	4	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	4	3	2	1
0	1	1	2	3	2	2	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	2	2	3	2	1	1
0	0	1	1	1	1	1	1	1	1	2	2	1	1	0	0	0	1	1	2	2	1	1	1	1	1	1	1	1	0
0	0	0	0	0	1	1	1	2	3	3	3	3	2	1	1	1	2	2	3	2	2	2	1	1	1	0	0	0	0
0	0	0	0	0	0	0	1	2	4	5	5	4	2	1	1	1	2	3	3	3	2	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	3	4	5	5	4	3	2	1	1	2	3	3	3	3	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	2	4	5	5	4	3	1	1	1	2	3	3	3	3	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	2	3	3	3	2	1	1	1	2	2	2	2	2	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	2	1	1	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0

Vertical axis: 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16

Horizontal axis: 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15

PROBABILITY OF DETECTION = .468 (Cumulative)

There are two main problems which need further work. The first is to incorporate a more realistic detection function, which is better suited to the sonobuoy problem. The second is to produce algorithms which will allow the user to pick rectangles, which are to stay in place for more than one time period.

As a step toward solving the second problem, we consider the problem of finding optimal plans which are restricted to have a fixed allocation for τ hours before choosing another allocation for the next τ hours.

As in Chapter II we let Ω be the set of possible paths of a target moving in discrete space. We assume that the searcher has τT time periods to search and let J be the set of all possible cells through which the paths pass. Because of the restriction mentioned above, a search plan is a function $\psi : J \times \{1,\ldots,T\} \to [0,\infty)$ such that

$$\psi(j,t) = \text{effort placed in cell } j \text{ at times } \tau(t-1)+1,\ldots,\tau t.$$

If a search plan ψ also satisfies

$$\sum_{j\in J} \psi(j,t) = m(t) \quad \text{for } t = 1,\ldots,T, \tag{III-1}$$

then we say $\psi \in \Psi(m)$. Note that we have $m(t)$ amount of effort available for each of the times $\tau(t-1)+1,\ldots,\tau t$. The only restriction is that we must allocate the effort in the same way for each of these time periods.

Given a search plan ψ we assume that the detection function is exponential so that the probability of detection using ψ is given by

$$P_{\tau T}[\psi] = 1 - \sum_{\omega\in\Omega} q(\omega) \exp\left\{-\sum_{t=1}^{T} \sum_{i=0}^{\tau-1} W(\omega_{\tau t-i})\, \psi(\omega_{\tau t-i}, t)/A(\omega_{\tau t-i})\right\},$$

where $q(\omega)$ is the probability of the path ω.

The problem is to maximize $P_{\tau T}[\psi]$ subject to (III-1).

Now let $\hat{J}^{\tau} = \{\hat{j} = (j_1, \ldots, j_\tau) \mid j_i \in J \text{ for } i = 1, \ldots, \tau \text{ and } j_1 \leq j_2 \ldots \leq j_\tau\}$. For any $\hat{j} \in \hat{J}^{\tau}$, we let

$$\Omega_t(\hat{j}) = \left\{ \omega \in \Omega \,\middle|\, \begin{array}{c} \omega \text{ passes through the cells in } \hat{j} \text{ in } \underline{\text{any}} \text{ order} \\ \text{during times } \tau(t-1)+1, \ldots, \tau. \end{array} \right\}.$$

For any search plan ψ, $\hat{j} \in \hat{J}^{\tau}$ and t define

$$\nu(\hat{j}, t, \psi) = \sum_{\omega \in \Omega_t(\hat{j})} q(\omega) \exp\{-\sum_{s \neq t} \sum_{i=0}^{\tau-1} W(\omega_{\tau s-i}) \; \psi(\omega_{\tau s-i}, s)/A(\omega_{\tau s-i})\}.$$

Then for each t, $1 \leq t \leq T$, we may write

$$P_{\tau T}[\psi] = 1 - \sum_{\hat{j} \in \hat{J}^{\tau}} \nu(\hat{j}, t, \psi) \exp\{-\sum_{i=0}^{\tau-1} W(j_i) \; \psi(j_i, t)/A(j_i)\}.$$

For a search plan ψ, we let ψ_t be given by

$$\psi_t(j) = \psi(j, t) \text{ for } j \in J.$$

For a search plan ψ and $1 \leq t \leq T$ we define

$$Q_t[\psi, f] = 1 - \sum_{\hat{j} \in \hat{J}^{\tau}} \nu(\hat{j}, t, \psi) \exp\{-\sum_{i=0}^{\tau-1} W(j_i, t) \; f(j_i)\}$$

where $f: J \to [0, \infty)$.

We can now characterize an optimal search plan $\psi^* \in \Psi(m)$.

THEOREM: $\psi^* \in \Psi(m)$ <u>maximizes</u> $P_{\tau T}$ <u>over</u> $\psi \in \Psi(m)$ <u>if and only if for each</u> t, $1 \leq t \leq T$,

$$Q_t[\psi^*, \psi_t^*] = \max\{ Q_t[\psi^*, f]: \; \Sigma_{j \in J} f(j) = m(t)\}. \qquad \text{(III-2)}$$

Proof. Since both $P_{\tau T}$ and $Q_t[\psi^*, \cdot]$ are concave functions and the effort constraints are linear, the Kuhn-Tucker conditions are necessary and sufficient for ψ^* to maximize $P_{\tau T}$ over $\psi \in \Psi(m)$ as well for ψ^*_t to satisfy (III-2). In fact, the Kuhn-Tucker conditions are the same for both problems, and thus we obtain the equivalence stated above. This proves the theorem.

Because of the above theorem, it is clear that an algorithm for computing an optimal plan restricted to fixing effort every τ times units can be obtained by following a procedure similar to that given in Chapter II when $\tau = 1$. The only difference is that maximizing $Q_t[\psi^*, \cdot]$ is not a standard stationary target problem as in the case $\tau = 1$. As noted above, $Q_t[\psi^*, \cdot]$ is a concave function and the constraint in (III-2) is a linear constraint; thus, we can apply a nonlinear programming approach to solving this problem.

TABLE III-6

COMPARISON OF RECTANGULAR PLAN WITH THE OPTIMAL PLAN FOR RADIAL FLEE

	Probability of Detection for	
Time	Rectangular Plan	5-Interval Optimal Plan
1	.329	.357
2	.411	.435
3	.437	.465
4	.454	.487
5	.468	.505

CHAPTER IV

ARBITRARY DISCRETE-TIME TARGET MOTION

In this chapter we consider optimal search for a target with arbitrary motion in discrete time. We discuss and give examples of a method for finding T-optimal search plans under the following conditions:

(1) The target motion is modeled by a discrete time stochastic process that can be simulated on a computer. This process may take place in continuous or discrete space.

(2) At each time t, m(t) effort is to be applied to searching for the target.

(3) The detection function is exponential but may vary over space.

(4) A grid is specified and allocations are required to be uniform within the cells of the grid.

Observe that the class of allowable target motions is very broad. For example, the target is not restricted to move among a set of cells as is usually the case when one deals with Markov chain models of target motion (as, for example, in Chapter II). The motion process need not be Markovian or even a mixture of Markovian processes. The process can be Gaussian, a constrained diffusion, or a random movement through a network. In fact, the optimizer described here could be coupled with the COMPASS, MEDSEARCH, or TARDIST programs to find optimal allocations of search effort over any interval of interest for any target motion processes produced by these programs.

The algorithm proceeds by obtaining a Monte Carlo sample of target paths and then optimizing over that sample. Thus the accuracy of the optimization will be

related to the accuracy with which the sample represents the target motion process. In addition the algorithm considers only search allocations which, during any one time period, are constant within the cells of a grid chosen by the user. However, this restriction is not equivalent to assuming that the target motion takes place in discrete space. The algorithm is based on the necessary and sufficient conditions stated in Chapter I.

While the algorithm described applies to a more general class of motion processes than the one described in Chapter II, it does, in effect, optimize over a finite sample of the target motion process, whereas the algorithm in Chapter II treats the target motion exactly. Thus in the case where a mixture of a small number of discrete time and space Markov processes provides a good representation of the target motion, the algorithm in Chapter II will be more accurate and faster.

In the first section we give examples of optimal plans computed by the algorithm which is described in the second section. In the third section we prove the algorithm converges to an optimal plan.

Examples

In this section we present three examples of optimal plans found using the algorithm described in the second section.

Example 1: Radial flee from a normal distribution. For this example we assume that the target has been detected by a sensor with poor localization so that its initial distribution is circular normal with standard deviation 20 miles in any direction. The target is known to be moving at 10 knots on a constant but unknown course. We assume that the course is chosen from a uniform distribution on 0° to 360°.

This target motion is an example of the classic fleeing target motion given in reference [i], page 17. That reference gives the probability distribution p_t for the target's location at time t given no search. In Chapter 7 of reference [i] reasonable search plans are suggested for this problem but in the years since that reference was prepared, no one has been able to find optimal plans for this problem.

We assume that the searcher, an aircraft, does not arrive on scene until hour 4. The dete-'.on function is given by

$$1 - e^{-Wz/A(j)},$$

where z is the number of sonobuoy hours placed in the cell j and W = 270 $(mi)^2$/sonobuoy-hour. This detection function approximates the probability of detection that one obtains from search by sonobuoys and is based on fitting an exponential function to a simulation of submarine detections by sonobuoy fields as discussed in reference [r]. The constant W = 270 $(mi)^2$/sonobuoy-hour, corresponds to good sonar conditions. We assume that 16 sonobuoys are available and that the search continues for 4 hours. Because of the time required to place the sonobuoys in the water, we assume that on the average only 16 x .75 sonobuoys are available for each hour of search. Since the constant W = 270 $(mi)^2$/sonobuoy-hour is the same over all space, we shall simply absorb this into the effort available at each hour and assume that there are 3240 units of effort available each hour.

Table IV-1 shows the myopic search plan for this problem for hours 4 through 7. The target distributions shown for each hour are the distributions at that time given failure of the previous search effort to detect the target. Observe that at time 4 the myopic allocation concentrates its effort most heavily in the center cells of the

TABLE IV-1

RADIAL FLEE FROM A NORMAL DISTRIBUTION

Myopic Plan -- Hour 4

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	0	2	6	1	1	0	0
1° N	0	1	2	23	32	35	21	5	0	0
	0	5	43	113	209	196	122	39	7	0
	2	21	129	281	452	408	313	117	25	2
0°	4	39	200	406	524	531	447	205	34	2
	2	40	210	395	509	531	425	182	29	5
	2	23	128	315	458	425	271	128	19	0
1° S	0	3	39	109	216	200	113	46	9	1
	0	0	7	24	40	43	30	8	0	0
	0	0	0	2	2	3	3	0	0	0
		1° W			0°				1° E	

MYOPIC ALLOCATION

	52	242	201	95
	199	301	307	238
0°	188	290	307	218
	98	248	218	38
		0°		

PROBABILITY OF DETECTION BY THE END OF TIME 4 = 0.274

TABLE IV-1 (Continued)

Myopic Plan -- Hour 5

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	4	6	10	8	7	1	0	0	0
	0	0	2	9	52	74	77	44	12	2	0	0
1° N	0	0	19	8.1	142	205	194	160	74	22	4	0
	0	5	55	155	215	184	182	205	156	54	5	0
	1	14	80	190	182	125	128	186	191	75	9	0
0°												
	0	11	81	193	168	132	127	182	212	65	12	1
	0	6	60	164	199	185	186	205	153	45	0	1
	0	1	15	77	161	216	198	155	75	27	1	0
1° S	0	0	0	18	55	75	92	40	19	1	0	0
	0	0	0	2	6	9	7	10	1	0	0	0
			1°W			0°			1°E			

MYOPIC ALLOCATION

	0	16	164	142	64	0
	52	182	121	115	164	54
	133	117	0	0	124	136
0°						
	141	84	0	0	117	176
	74	152	123	125	164	46
	0	68	184	149	51	0
			0°			

PROBABILITY OF DETECTION BY THE END OF TIME 5 = 0.406

TABLE IV-1 (Continued)

Myopic Plan -- Hour 6

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	0	0	2	4	2	0	0	0	0
	0	0	1	8	17	21	20	17	6	2	0	0
	0	1	9	38	79	118	100	87	36	7	4	0
1^o N	0	8	45	96	139	102	114	105	104	50	7	0
	2	15	89	124	95	91	89	106	117	80	13	2
	2	30	111	110	86	61	60	92	101	106	22	1
0^o												
	2	23	116	119	85	58	54	93	119	101	26	3
	1	20	94	132	93	84	96	97	117	70	16	1
1^o S	0	3	35	96	134	112	118	124	110	41	7	1
	0	0	14	43	73	103	102	71	44	14	0	0
	0	0	0	7	23	30	33	20	6	2	0	0
	0	0	0	0	2	2	1	1	0	0	0	0

1^oW 0^o 1^oE

MYOPIC ALLOCATION

1^oN	0	0	0	123	56	0	0	0
	0	41	189	64	110	77	72	0
	11	143	38	20	13	83	122	0
	99	94	0	0	0	25	62	81
0^o								
	116	128	0	0	0	30	129	62
	31	170	28	0	41	46	121	0
1^o S	0	41	174	104	124	142	95	0
	0	0	0	69	65	0	0	0

1^oW 0^o 1^oE

PROBABILITY OF DETECTION BY THE END OF TIME 6 = 0.436

TABLE IV-1 (Continued)

Myopic Plan -- Hour 7

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^5

	0	0	10	30	40	90	50	70	0	10	0	0	0
	0	0	0	250	282	494	416	415	190	30	10	0	0
	0	60	236	590	830	826	784	846	628	280	70	0	0
1°N	20	159	746	803	707	752	681	718	825	678	210	10	0
	60	422	802	785	644	558	517	649	654	805	369	50	0
	90	507	735	704	520	181	211	531	778	776	429	80	0
0°													
	100	510	830	692	501	167	179	528	687	792	545	90	10
	40	381	729	799	597	526	543	616	706	733	339	30	0
1°S	10	235	733	769	662	792	693	729	845	610	190	10	0
	0	10	206	657	768	883	811	836	560	370	60	0	0
	0	0	60	120	491	401	556	374	200	30	0	0	0
	0	0	0	10	70	70	90	60	10	0	0	0	0

1°W   0°   1°E   2°E

MYOPIC ALLOCATION

1°N	0	0	117	115	94	125	5	0
	74	104	53	77	38	59	114	36
	103	95	15	0	0	18	22	105
	68	51	0	0	0	0	91	90
0°								
	117	44	0	0	0	0	41	98
	65	102	0	0	0	0	52	67
1°S	67	86	26	98	45	65	124	0
	0	23	86	141	108	120	0	0

1°W   0°   1°E

PROBABILITY OF DETECTION BY THE END OF TIME 7 = 0.543

target distribution, but at times 5 through 7 effort is concentrated in an annulus centered at the mean of the target distribution. Except for variations caused by sampling error in the underlying target distribution, the effort distribution is seen to be fairly symmetric. Table IV-2 shows the optimal plan for this example. For hours 4-7 the cells in which the optimal plan places a larger amount of effort than the myopic plan are indicated by shading. Notice that the optimal plan places substantially more (e.g., 51 percent more) effort in the center four cells at hour 4 than the myopic plan. At times 5-7 the optimal plan appears to be more concentrated than the myopic plan. In addition one can see that at time 5, the optimal plan chooses to allocate effort heavily to the eastern side of the distribution and then balances this by heavy allocations on the western side at times 6 and 7.

Although the optimal plan is qualitatively quite different from the myopic plan, one can see from Table IV-3 that there is very little difference in the detection probabilities from these two plans.

<u>Example 2: Radial flee from a uniform distribution.</u> In this example the search assumptions are exactly as in Example 1 except that the target's initial distribution is uniform over a square 80 miles on a side. Table IV-4 shows the myopic plan and Table IV-5 shows the optimal plan for this example.

Since the initial distribution is uniform rather than normal, one might have expected that both the optimal and myopic plans would stop searching in the center of the distribution earlier than in Example 1. However, by comparing the myopic and optimal allocations at time 5 for this example to those in the previous example, one can see that both plans and in particular the optimal plan continue to search the center of the distribution longer when the initial distribution is uniform than

TABLE IV-2

RADIAL FLEE FROM A NORMAL DISTRIBUTION

Optimal Plan -- Hour 4

Note: Shading indicates the cells in which the optimal plan places more effort than the myopic plan.

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	0	2	6	1	1	0	0
1°N	0	1	2	23	32	35	21	5	0	0
	0	5	43	113	209	196	122	39	7	0
	2	21	129	281	452	408	313	117	25	2
0°	4	39	200	406	524	531	447	205	34	2
	2	40	210	395	509	531	425	182	29	5
	2	23	128	315	458	425	271	128	19	0
1°S	0	3	39	109	216	200	113	46	9	1
	0	0	7	24	40	43	30	8	0	0
	0	0	0	2	2	3	3	0	0	0
		1°W			0°			1°E		

OPTIMAL ALLOCATION

	0	206	178	70
0°	152	456	449	218
	128	457	456	166
	0	153	151	0
		0°		

PROBABILITY OF DETECTION BY THE END OF TIME 4 = 0.265

TABLE IV-2 (Continued)

Optimal Plan -- Hour 5

Note: Shading indicates the cells in which the optimal plan places more effort than the myopic plan.

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	4	6	10	8	7	1	0	0	0
	0	0	2	9	52	74	77	44	12	2	0	0
1°N	0	0	19	83	150	213	199	163	75	22	4	0
	0	5	55	163	237	176	174	214	160	54	5	0
	1	14	80	199	180	90	94	173	195	75	9	0
0°												
	0	11	81	207	170	90	90	181	222	65	12	1
	0	6	60	177	243	196	192	225	159	45	0	1
1°S	0	1	15	81	183	235	213	162	76	27	1	0
	0	0	0	18	55	75	92	40	19	1	0	0
	0	0	0	2	6	9	7	10	1	0	0	0

1°W 0° 1°E

OPTIMAL ALLOCATION

	0	0	75	108	133	0
	19	184	189	188	253	103
	29	158	0	0	208	161
0°						
	0	103	0	0	184	167
	28	157	180	187	206	122
	0	38	53	25	33	0

0°

PROBABILITY OF DETECTION BY THE END OF TIME 5 = 0.395

TABLE IV-2 (Continued)

Optimal Plan -- Hour 6

Note: Shading indicates the cells in which the optimal plan allocates more effort than the myopic plan.

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	0	0	2	4	2	0	0	0	0
	0	0	1	8	17	21	20	17	6	2	0	0
	0	1	9	38	83	131	105	80	35	7	4	0
1°N	0	8	46	105	157	115	119	93	98	49	7	0
	2	15	94	146	87	65	65	83	107	75	13	2
	2	30	128	135	67	44	43	63	94	103	22	1
0°												
	2	23	137	164	63	42	39	68	123	102	26	3
	1	20	102	169	94	62	70	86	118	68	16	1
1°S	0	3	36	113	174	153	157	138	112	41	7	1
	0	0	14	44	76	124	121	74	44	14	0	0
	0	0	0	7	23	30	33	20	6	2	0	0
	0	0	0	0	2	2	1	1	0	0	0	0
			1°W				0°			1°E		

OPTIMAL ALLOCATION

	0	0	0	117	45	0	0	0
1°N								
	0	34	194	111	111	19	58	0
	0	130	0	0	0	14	90	0
	125	155	0	0	0	0	53	75
0°								
	126	199	0	0	0	0	148	67
	47	203	32	0	0	6	21	0
	0	80	264	162	180	134	100	0
1°S								
	0	0	0	29	53	0	0	0
	1°W			0°			1°E	

PROBABILITY OF DETECTION BY THE END OF TIME 6 = 0.488

TABLE IV-2 (Continued)

Optimal Plan -- Hour 7

Note: Shading indicates the cells in which the optimal plan allocates more effort than the myopic plan.

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

0	0	1	3	4	9	5	7	0	1	0	0	0
0	0	0	25	28	50	42	42	19	3	1	0	0
0	6	24	61	93	102	89	83	59	28	7	0	0
2	16	79	93	75	64	58	69	79	66	21	1	0
6	43	95	85	54	41	38	52	61	77	37	5	0
9	49	88	60	39	15	17	39	64	76	43	8	0
10	50	109	61	36	14	13	40	58	80	54	9	1
4	37	86	89	47	38	41	51	70	73	34	3	0
1	23	76	86	67	69	62	76	88	60	19	1	0
0	1	20	67	87	125	110	96	58	37	6	0	0
0	0	6	12	50	42	57	38	20	3	0	0	0
0	0	0	1	7	7	9	6	1	0	0	0	0

OPTIMAL ALLOCATION

0	0	145	180	128	100	0	0
80	146	57	0	0	26	81	6
152	110	0	0	0	0	0	71
124	0	0	0	0	0	0	66
206	0	0	0	0	0	0	86
111	126	0	0	0	0	32	46
64	111	15	26	0	66	120	0
0	11	118	263	210	158	0	0

PROBABILITY OF DETECTION BY THE END OF TIME 7 = 0.551

TABLE IV-3

PROBABILITY OF DETECTION FOR OPTIMAL AND MYOPIC PLANS FOR RADIAL FLEE FROM A NORMAL DISTRIBUTION

	PROBABILITY OF DETECTION	
HOUR	OPTIMAL	MYOPIC
1	0.000	0.000
2	0.000	0.000
3	0.000	0.000
4	0.265	0.274
5	0.395	0.406
6	0.488	0.486
7	0.551	0.543

TABLE IV-4

RADIAL FLEE FROM A UNIFORM DISTRIBUTION

Myopic Plan -- Hour 4

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	8	25	53	73	47	5	0
1°N	16	93	157	226	248	163	80	15
	44	141	195	322	328	231	173	30
	55	218	365	522	520	338	227	66
0°								
	48	208	343	511	513	344	188	61
	51	155	213	331	395	245	149	44
1°S	13	72	173	245	225	153	92	15
	0	12	42	50	59	54	12	0
	1°W			0°			1°E	

MYOPIC ALLOCATION

	0	0	23	61	0	0
	0	0	165	172	32	0
	9	215	358	357	184	25
0°						
	0	190	350	351	191	0
	0	0	176	247	56	0
	0	0	56	22	0	0
			0°			

PROBABILITY OF DETECTION BY THE END OF TIME 4 = 0.243

TABLE IV-4 (Continued)

Myopic Plan -- Hour 5

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

1°N	10	44	73	124	154	104	30	6
	52	83	152	183	165	141	116	42
	85	137	147	161	152	162	160	93
	118	166	154	148	146	160	168	121
0°	116	186	156	156	152	157	165	129
	106	144	149	157	151	164	144	85
1°S	42	99	161	169	159	156	109	54
	2	44	93	124	132	94	28	5
	1°W			0°			1°E	

MYOPIC ALLOCATION

1°N	0	0	1	89	0	0	0
	0	83	156	115	54	0	0
	42	70	107	84	108	103	0
	119	87	73	66	105	123	0
0°	164	94	95	84	96	115	16
	60	76	97	81	113	63	0
1°S	0	106	171	101	95	0	0
	0	0	3	25	0	0	0
			0°			1°E	

PROBABILITY OF DETECTION BY THE END OF TIME 5 = 0.358

TABLE IV-4 (Continued)

Myopic Plan -- Hour 6

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	1	7	19	41	53	35	8	0	0
1°N	0	38	66	94	126	112	108	65	22	1
	11	59	100	99	95	101	103	103	72	12
	38	95	98	95	82	88	94	111	96	29
0°	34	117	102	92	67	73	81	88	130	38
	38	110	98	74	73	73	83	94	118	50
	40	107	101	90	80	87	93	105	86	33
1°S	12	62	109	108	102	95	101	109	89	18
	0	27	64	93	128	118	85	66	24	2
	0	0	10	33	41	44	44	7	1	0

1°W 0° 1°E

MYOPIC ALLOCATION

1°N	0	0	45	161	113	100	0	0
	0	70	66	49	73	83	81	0
	51	61	49	0	17	45	112	54
0°	132	79	34	0	0	0	18	175
	107	61	0	0	0	0	46	134
	96	71	29	0	13	38	90	8
1°S	0	103	99	78	51	75	104	22
	0	0	42	168	135	4	0	0

1°W 0° 1°E

PROBABILITY OF DETECTION BY THE END OF TIME 6 = 0.435

TABLE IV-4 (Continued)

Myopic Plan -- Hour 7

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^5

	0	70	299	482	743	959	679	343	50	0
	150	522	700	864	635	719	749	729	540	100
1°N	325	613	744	721	709	632	643	646	754	312
	704	787	656	727	576	533	797	762	615	640
0°	813	618	672	516	214	241	556	720	728	771
	616	705	634	542	219	227	518	605	717	827
	747	819	706	767	502	626	749	765	783	649
1°S	344	760	691	749	596	615	774	670	697	435
	90	466	695	761	741	612	775	751	591	108
	0	70	410	551	769	839	751	280	90	0

1°W 0° 1°E

MYOPIC ALLOCATION

	0	0	0	0	66	168	30	0	0	0
1°N	0	0	42	126	3	53	69	58	0	0
	0	0	66	54	47	1	8	10	71	0
	44	88	16	57	0	0	94	75	0	6
0°	102	0	25	0	0	0	0	53	57	80
	0	45	2	0	0	0	0	0	51	108
	68	105	45	78	0	0	69	77	87	11
1°S	0	75	40	69	0	0	82	24	40	0
	0	0	39	75	65	0	83	70	0	0
	0	0	0	0	80	114	70	0	0	0

1°W 0° 1°E

PROBABILITY OF DETECTION BY THE END OF TIME 7 = 0.492

TABLE IV-5

RADIAL FLEE FROM A UNIFORM DISTRIBUTION

Optimal Plan -- Hour 4

Note: Shading indicates the cells in which the optimal plan places more effort than the myopic plan.

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	8	25	53	73	47	5	0
1^oN	16	93	157	226	248	163	80	15
	44	141	195	322	328	231	173	30
	55	218	365	522	520	338	227	66
0^o								
	48	208	343	511	513	344	188	61
	51	155	213	331	395	245	149	44
	13	72	173	245	225	153	92	15
1^oS	0	12	42	50	59	54	12	0

1^oW 0^o 1^oE

OPTIMAL ALLOCATION

	0	0	0	4	0
	0	0	171	179	0
	2	236	416	414	188
0^o					
	0	201	402	401	193
	0	0	169	253	11

0^o

PROBABILITY OF DETECTION BY THE END OF TIME 4 = 0.242

TABLE IV-5 (Continued)

Optimal Plan -- Hour 5

Note: **Shading indicates the cells in which the optimal plan places more effort than the myopic plan.**

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	10	44	74	129	164	106	30	6
1^oN	52	83	153	190	178	148	117	42
	85	137	144	154	146	168	166	93
	119	166	144	131	129	155	174	125
0^o								
	116	185	148	139	135	152	166	129
	106	143	148	154	147	174	150	85
	42	99	165	206	167	163	110	54
1^oS	2	44	94	134	136	94	28	5
		1^oW			0^o		1^oE	

OPTIMAL ALLOCATION

	0	59	103	77	0	5
	61	115	161	134	118	86
	98	137	125	117	167	78
0^o						
	127	147	149	140	158	61
	0	73	148	131	137	46
	0	47	98	55	84	0
				0^o		

PROBABILITY OF DETECTION BY THE END OF TIME 5 = 0.351

TABLE IV-5 (Continued)

Optimal Plan -- Hour 6

Note: **Shading indicates the cells in which the optimal plan places more effort than the myopic plan.**

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	1	7	19	41	65	35	8	0	0
	0	38	66	99	144	144	122	66	22	1
1°N	11	58	100	101	98	106	120	109	73	12
	38	94	92	80	66	71	84	120	100	29
	34	121	99	74	52	57	65	89	143	38
0°										
	38	114	97	60	57	57	66	98	129	52
	40	114	110	82	66	71	82	113	89	33
	12	63	116	123	111	101	110	115	89	18
1°S	0	27	65	104	153	136	89	66	24	2
	0	0	10	33	41	46	44	7	1	0

1°W 0° 1°E

OPTIMAL ALLOCATION

1°N	0	0	4	180	106	112	0	0
	0	69	21	68	88	139	118	0
	31	37	6	0	0	21	159	97
	138	96	0	0	0	0	25	174
0°								
	114	59	0	0	0	0	40	101
	43	57	0	0	0	0	95	0
1°S	0	100	107	90	71	68	140	59
	0	0	58	184	152	0	0	0

1°W 0° 1°E

PROBABILITY OF DETECTION BY THE END OF TIME 6 = 0.435

TABLE IV-5 (Continued)

Optimal Plan -- Hour 7

Note: Shading indicates the cells in which the optimal plan places more effort than the myopic plan.

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	7	30	51	78	119	71	34	5	0
1°N	15	52	74	104	73	83	86	74	53	10
	33	61	75	72	62	55	62	64	74	31
	72	81	63	65	45	43	68	73	62	61
0°	81	63	59	42	17	19	44	63	81	79
	61	75	56	43	17	18	41	53	86	89
	79	98	72	66	39	50	65	76	87	66
	35	83	78	77	52	54	79	68	67	41
1°S	9	47	75	90	87	71	87	76	57	10
	0	7	41	55	80	89	77	28	9	0

1°W 0° 1°E

OPTIMAL ALLOCATION

	0	0	0	0	68	239	34	0	0	0
1°N	0	0	49	183	43	95	107	51	0	0
	0	0	54	36	0	0	0	0	47	0
	38	83	0	0	0	0	16	40	0	0
0°	85	0	0	0	0	0	0	0	87	73
	0	53	0	0	0	0	0	0	109	121
	74	160	37	2	0	0	0	61	113	3
1°S	0	94	67	66	0	0	73	17	7	0
	0	0	51	126	111	30	115	57	0	0
	0	0	0	0	79	121	63	0	0	0

1°W 0° 1°E

PROBABILITY OF DETECTION BY THE END OF TIME 7 = 0.496

when it is normal. Table IV-6 shows that again there is surprisingly little difference between the detection probabilities for the myopic and optimal plans.

Example 3: Two-scenario target motion. In this example the target's initial distribution at the beginning of hour 1 is circular normal with standard deviation 10 miles in any direction. There are two possible scenarios for the target's motion; both are equally likely. For scenario 1 the target is assumed to be traveling at a speed uniformly distributed between 15 and 20 knots. The target's course is uniformly distributed between 150° and 210°. The target maintains a constant course and speed chosen from these distributions until hour 5 when it makes an independent draw from a distribution on speed which is uniform from 10-20 knots and an independent choice of course from a distribution which is uniform from $120-240^{\circ}$. The target continues at this course and speed for the remainder of the problem.

In scenario 2 the target is assumed to be traveling at 15 knots and to make a draw from a truncated triangular distribution (see Figure II-1) with mean course 75°, maximum course 105°, best course 90°, and weight factor 2. The target maintains this course and speed until time 5 when it makes an independent draw from the same distribution. It retains this course and speed for the remainder of the problem.

As in the first two examples, we have 4 hours of VP search available beginning at hour 4 and we wish to maximize the probability of detection by the end of hour 7. The detection assumptions are similar to those which are discussed in Example 1. However, the sonar conditions vary from good ($W = 200$) to very poor ($W = 25$) in the manner shown in Figure IV-1. As before we assume that there are 12 sonobuoy hours available per hour for each time period.

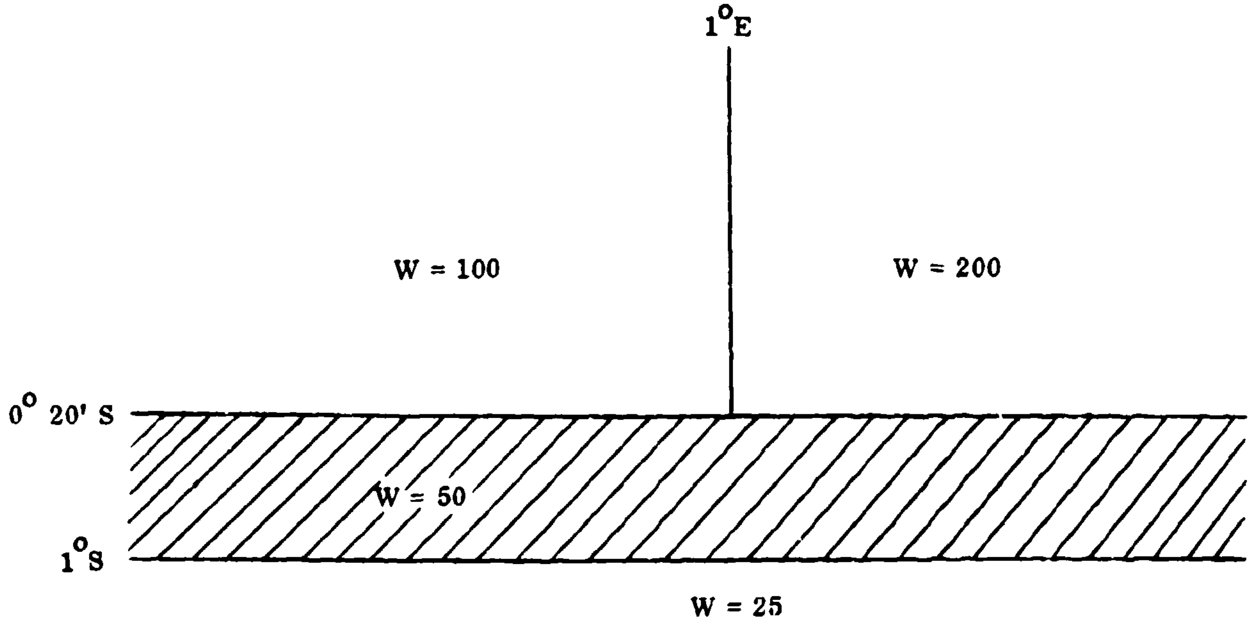
TABLE IV-6

PROBABILITY OF DETECTION FOR THE OPTIMAL AND MYOPIC PLANS FOR RADIAL FLEE FROM A UNIFORM DISTRIBUTION

	PROBABILITY OF DETECTION	
HOUR	OPTIMAL	MYOPIC
1	0.000	0.000
2	0.000	0.000
3	0.000	0.000
4	0.242	0.243
5	0.351	0.358
6	0.435	0.435
7	0.496	0.492

FIGURE IV-1

SONAR REGIONS FOR EXAMPLE 3



Tables IV-7 and IV-8 show the myopic and optimal plans for this example. At hours 4 and 5 the myopic plan concentrates its effort on scenario 2, the eastward moving one. This is because most of its probability mass is located in areas of relatively good sonar conditions. Even though by hours 6 and 7 the first or south moving scenario has distinctly the higher probability given failure to detect during hours 4 and 5, the myopic plan continues to allocate most of the effort to scenario 2 because scenario 1 has now moved into the very poor sonar region. By contrast the optimal plan concentrates heavily on scenario 1 during hours 4 and 5 before that scenario moves into the region of poor sonar conditions. Then during hours 6 and 7 the optimal plan searches scenario 2 which has moved into a better sonar region.

Table IV-9 shows the detection probabilities for the optimal and myopic plan. Observe that at hour 4 the myopic plan has a substantially higher detection probability than the optimal plan but that by hour 7 the optimal plan has detection probability .58 versus .48 for the myopic, an increase of 19 percent over the myopic plan.

Smoothing. In the above examples a Monte Carlo simulation of 10,000 sample paths was used to represent the target motion process. Even with this rather large number of replications, the target location distributions still have considerable statistical variation. This indicates that a smoothing technique such as the one examined in reference [8] might be helpful. It is planned to investigate this possibility in future work.

TABLE IV-7

TWO SCENARIO EXAMPLE

Myopic Plan -- Hour 4

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

	0	0	0	0	6	14	1
	0	0	0	1	76	137	14
0°	0	0	1	1	72	141	13
	1	16	28	28	23	13	1
	2	48	110	110	46	3	0
1°S	0	11	37	31	12	0	0
	0	0	0	0	0	0	0

0° 1°E

MYOPIC ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

	0	0	156	393
0°	0	0	137	405
	0	0	0	0
	53	56	0	0

0°

PROBABILITY OF DETECTION BY THE END OF TIME 4 = 0.235

TABLE IV-7 (Continued)

Myopic Plan -- Hour 5

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

	0	0	0	0	0	3	98	114	3
0°	0	0	0	0	0	43	644	443	48
	0	0	0	0	0	36	651	436	47
	0	6	24	42	26	31	110	109	3
	0	48	353	570	576	337	45	0	0
1°S	1	37	371	836	831	380	40	0	0
	0	3	39	111	108	40	4	0	0

1°W 0° 1°E

MYOPIC ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

0°	0	0	0	338	233
	0	0	0	343	230
	0	0	0	0	0
	24	32	0	0	0

0°

PROBABILITY OF DETECTION BY THE END OF TIME 5 = 0.374

TABLE IV-7 (Continued)

Myopic Plan -- Hour 6

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	22	116	72	1
0°	0	0	0	0	0	1	74	306	119	13
	0	0	0	0	0	0	72	300	125	16
	0	2	1	4	4	3	17	124	67	0
1°S	1	53	105	134	138	97	37	3	0	0
	10	165	424	653	684	435	133	11	0	0
	5	50	279	450	472	261	57	7	0	0
	0	5	27	48	33	22	1	0	0	0

1°W 0° 1°E

MYOPIC ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

	0	0	0	0	88	0
0°	0	0	0	0	282	92
	0	0	0	0	278	103
	0	0	0	0	0	0
	0	0	0	0	0	0
1°S	141	216	0	0	0	0

0° 1°E

PROBABILITY OF DETECTION BY THE END OF TIME 6 = 0.447

<u>**TABLE IV-7 (Continued)**</u>

Myopic Plan -- Hour 7

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	0	0	0	0	0	1	4	0	0
	0	0	0	0	0	0	0	1	28	91	36	0
	0	0	0	0	0	0	0	1	97	90	63	2
0°	0	0	0	0	0	0	0	1	83	93	63	0
	0	0	1	1	0	0	0	1	44	119	33	1
1°S	0	6	20	21	25	26	30	20	5	1	1	0
	1	45	143	207	248	227	212	148	43	0	0	0
	6	43	219	386	491	481	394	210	39	1	0	0
	0	6	52	186	227	231	138	45	8	0	0	0
2°S	0	0	3	15	18	18	10	0	0	0	0	0

1°W 0° 1°E 2°E

MYOPIC ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

	0	0	0	0	0	115	0
	0	0	0	0	128	112	42
0°	0	0	0	0	95	119	42
	0	0	0	0	0	0	0
1°S	0	0	0	0	0	0	0
	0	0	0	0	0	0	0
	291	256	0	0	0	0	0

0° 1°E

PROBABILITY OF DETECTION BY THE END OF TIME 7 = 0.485

TABLE IV-8

TWO SCENARIO EXAMPLE

Optimal Plan -- Hour 4

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

	0	0	0	0	6	14	1
	0	0	0	1	76	137	14
0°	0	0	1	1	72	141	13
	1	16	28	28	23	13	1
1°S	2	48	110	110	46	3	0
	0	11	37	31	12	0	0
	0	0	0	0	0	0	0
				0°		1°E	

OPTIMAL ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

1°S	590	610
	0°	

PROBABILITY OF DETECTION BY THE END OF TIME 4 = 0.116

TABLE IV-8 (Continued)

Optimal Plan -- Hour 5

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

	0	0	0	0	0	0	10	13	0
	0	0	0	0	0	6	113	101	5
0°	0	0	0	0	0	5	113	104	5
	0	1	2	4	3	3	12	12	0
1°S	0	5	31	43	43	30	4	0	0
	0	4	32	57	55	34	4	0	0
	0	0	4	11	11	4	0	0	0

1°W 0° 1°E

OPTIMAL ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

	0	0	0	0	0	127
0°	0	0	0	0	0	164
	0	0	0	0	0	0
1°S	100	344	348	57	0	0
	0	59	0	0	0	0

0°

PROBABILITY OF DETECTION BY THE END OF TIME 5 = 0.260

TABLE IV 8 (Continued)

Optimal Plan -- Hour 6

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	2	15	8	0
0°	0	0	0	0	0	0	17	124	33	1
	0	0	0	0	0	0	17	118	30	2
	0	0	0	0	0	0	2	15	8	0
1°S	0	5	8	10	10	8	4	0	0	0
	1	15	27	37	38	30	12	1	0	0
	0	5	23	33	36	23	5	1	0	0
	0	0	3	5	3	2	0	0	0	0

1°W 0° 1°E

OPTIMAL ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

	22	0
0°	345	241
	358	233

1°E

PROBABILITY OF DETECTION BY THE END OF TIME 6 = 0.506

TABLE IV-8 (Continued)

Optimal Plan -- Hour 7

TARGET DISTRIBUTION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

	0	0	0	0	0	0	0	0	1	4	0	0
	0	0	0	0	0	0	0	1	41	154	38	0
	0	0	0	0	0	0	0	1	218	237	63	2
0°												
	0	0	0	0	0	0	0	1	189	214	60	0
	0	0	1	1	0	0	0	1	59	153	36	1
	0	6	19	18	21	24	26	19	5	1	1	0
1°S												
	1	42	106	122	163	147	125	108	40	0	0	0
	6	39	167	276	338	341	287	165	37	1	0	0
2°S	0	6	49	154	171	182	137	42	8	0	0	0
	0	0	3	15	18	18	10	0	0	0	0	0

1°W 0° 1°E 2°E

OPTIMAL ALLOCATION

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^2

	0	186	0
	255	272	8
0°			
	226	252	0

1°E

PROBABILITY OF DETECTION BY THE END OF TIME 7 = 0.577

TABLE IV-9

PROBABILITY OF DETECTION FOR OPTIMAL AND MYOPIC PLANS FOR TWO SCENARIO EXAMPLE

	PROBABILITY OF DETECTION	
HOUR	OPTIMAL	MYOPIC
1	0.000	0.000
2	0.000	0.000
3	0.000	0.000
4	0.116	0.235
5	0.256	0.374
6	0.506	0.447
7	0.577	0.486

Description of the Algorithm

We suppose that we are given N sample paths drawn from the process $\{X_t; t = 1, \ldots, T\}$. The n^{th} sample path is assumed to have the form $(x_1^n, x_2^n, \ldots, x_T^n)$, w_n, where x_s^n is the position of the target at time s in the n^{th} sample path and w_n is the sample probability that the n^{th} sample path represents the target's actual path (usually $w_n = 1/N$).

Conceptually the first act of the optimizer is to convert the sample paths into sequences of cell numbers. Thus, the n^{th} sample path becomes

$$(j_1^n, j_2^n, \ldots, j_T^n), \; w_n,$$

where $j_s^n = c(x_s^n)$ is the index of the cell into which x_s^n falls.

As in the third section of Chapter II we define the function Ξ_t which acts on a search plan ψ by replacing the allocation at time t, $\psi(\cdot, t)$, with f^*, an optimal allocation of m(t) effort for $g_\psi(\cdot, t)$, the posterior target distribution at time t given failure to detect at all times other than t using ψ. That is

$$\Xi_t \, \psi(j, s) = \begin{cases} \psi(j, s) & \text{for } s \neq t, \\ f^*(j) & \text{for } s = t. \end{cases}$$

<u>First pass.</u> Set $\psi_0(j, t) = 0$ for $j \in J$, $t = 1, \ldots, T$. We begin with $t = 1$. For each cell j we accumulate all the probabilities, w_n, of paths such that $j_1^n = j$ to compute $p_1(j)$, the probability that the target is in cell j at time 1 for $j = 1, \ldots, J$. That is

$$p_1(j) = \sum_{\{n: j_1^n = j\}} w_n. \qquad \text{(IV-1)}$$

Using a standard algorithm (e.g., the algorithm in Example 2.2.8 of reference [h], we calculate the allocation of effort f^* which maximizes the probability of detection,

$$\sum_{j=1}^{J} p_1(j)\left[1-\exp\left(-W(j)\, f^*(j)/A(j)\right)\right], \tag{IV-2}$$

for a stationary target with probability $p_1(j)$ of being in cell j for $j = 1, \ldots, J$, subject to

$$\sum_{j=1}^{J} f^*(j) = m(t). \tag{IV-3}$$

Let $\psi_1 = \Xi_1 \psi_0$ and compute

$$\tilde{w}_n = w_n \exp\left(-W(j_1^n)\, \psi_1(j_1^n, 1)/A(j)\right) \qquad \text{for } n = 1, \ldots, N. \tag{IV-4}$$

The value $\tilde{w}_n$ is the probability that the target is following the n^{th} sample path and is not detected by the search effort at time 1.

For $t = 2$, we compute $\tilde{g}_{\psi_1}(j, 2)$, the probability that the target is in cell j at time 2 and not detected by the search applied at time 1 by

$$\tilde{g}_{\psi_1}(j, 2) = \sum_{\{n: j_2^n = j\}} \tilde{w}_n \qquad \text{for } j = 1, \ldots, J. \tag{IV-5}$$

Let $g_{\psi_1}(\cdot, 2)$ be the probability distribution that one obtains by normalizing $\tilde{g}_{\psi_1}(\cdot, 2)$ so that it sums to 1. As above we find an optimal allocation of m(2) effort for the stationary target problem with distribution $\tilde{g}_{\psi_1}(\cdot, 2)$, and let $\psi_2 = \Xi_2 \psi_1$. The weights $\tilde{w}_n$ are then multiplied by $\exp(-W(j_2^n)\, \psi_2(j_2^n, 2)/A(j_2^n))$ to obtain the revised values of $\tilde{w}_n$ for $n = 1, \ldots, N$.

This process is continued for $t = 3, \ldots, T$. At the end of this first pass we have calculated the myopic plan, ψ_T, i.e., the plan which at each time period allocates its effort to maximize the probability of detection during that time period given failur to detect the target in previous time periods.

Second pass. Let w_n^1 be the value of $\tilde{w}_n$ obtained by the end of the first pass described above, i.e., w_n^1 is the probability that the target is following the n^{th} sample path and is not detected by the effort allocated in the first pass. For . we calculate

$$\hat{w}_n = w_n^1 \exp\left(W(j_1^n)\ \psi_T(j_1^n, 1)/A(j_1^n)\right) \qquad \text{for } n = 1, \ldots, N,$$

and accumulate $\hat{w}_n$ into cell j_1^n, to calculate $g_{\psi_T}(j, 1)$, the probability that the target is in cell j at time 1 given failure to detect at all times in the future. Generally $p_t \neq g_{\psi_T}(\cdot, 1)$.

We then find an optimal allocation of m(1) effort for the distribution $g_{\psi_T}(\cdot, 1)$, set $\psi_{T+1} = \Xi\, \psi_T$, and compute new values for $\tilde{w}_n$ by

$$\tilde{w}_n = \hat{w}_n \exp\left(-W(j_1^n)\ \psi_{T+1}(j_1^n, 1)/A(j_1^n)\right).$$

For t = 2 we calculate

$$\hat{w}_n = \tilde{w}_n \exp\left(W(j_2^n)\ \psi_1(j_2^n, w)/A(j_2^n)\right),$$

and repeat the above process for time 2. This continues for t = 3, ..., T and the second pass is completed. Additional passes proceed in a similar fashion.

Observe that at the end of the l^{th} pass

$$1 - P_T[\psi_{lT}] = \sum_{n=1}^{N} \tilde{w}_n, \qquad \text{(IV-6)}$$

and that $1 - P_T[\psi_{lT}]$ is monotonically decreasing in l. We shall show in the third section that as $l \to \infty$, there is a convergent subsequence of plans $\{\psi_{l_i T}\}_{i=1}^{\infty}$ such that $\psi^*(j, t) \equiv \lim_{i \to \infty} \psi_{l_i T}(j, t)$ exists for j = 1, ..., J and t = 1, ..., T and such that

ψ^* is T-optimal. Furthermore,

$$\lim_{l \to \infty} P_T[\psi_{lT}] = P_T[\psi^*]. \qquad \text{(IV-7)}$$

Thus one can come as close to the optimal plan as he wishes by making enough passes. For computational purposes, one usually chooses an $\epsilon > 0$ and stops when $P_T[\psi_{lT}] - P_T[\psi_{(l-1)T}] < \epsilon$.

The computer program which implements this algorithm is described in reference [g].

Proof of Convergence

In this section we take the point of view that a search plan ψ is a density defined on $Y \times \{1, \ldots, T\}$ where Y is the plane. Then $\Psi(m)$ becomes the class of plans ψ such that

$$\int_Y \psi(y, t)\, dy = m(t) \qquad \text{for } t = 1, \ldots, T.$$

The search plans that we consider in this chapter are a subclass $\hat{\Psi}(m)$ of $\Psi(m)$ in which the effort density is constant over the grid cells. Thus we can identify a plan ψ obtained in the above algorithm with a member $\hat{\psi} \in \hat{\Psi}(m)$ as follows: Let $c(y)$ be the cell containing the point y. Then

$$\hat{\psi}(y, t) = \frac{\psi(c(y), t)}{A(c(y))} \qquad \text{for } y \in Y,\ t = 1, \ldots, T.$$

In order to show that the above algorithm converges to a plan $\psi^* \in \hat{\Psi}(m)$ (under the above identification) such that

$$P_T[\psi^*] = \max\{ P_T[\hat{\psi}] : \hat{\psi} \in \hat{\Psi}(m)\},$$

we first find necessary and sufficient conditions for $\psi^* \epsilon \hat{\Psi}(m)$ to be T-optimal within $\hat{\Psi}(m)$, i.e., for ψ^* to satisfy the above equality. For this proof we shall begin by taking the point of view that there is a regular detection function b (i.e., b has a positive continuous and strictly decreasing derivative b') such that $b(\Sigma_{t=1}^{T} \psi(X_t(\omega), t))$ is the probability of detecting the target given it follows the path ω. Similarly

$$P_T[\psi] = E\left[b\left(\sum_{t=1}^{T} \psi(X_{t_1}(\omega), t)\right)\right].$$

Let F be the set of real-valued Borel functions defined on $Y \times \{1, \ldots, T\}$ such that

$$\int_Y |f(y, t)| dy < \infty \qquad \text{for } t = 1, \ldots, T,$$

$$||f|| \equiv \sum_{t=1}^{T} \operatorname*{ess\,sup}_{y \epsilon Y} |f(y, t)| < \infty.$$

Let

$$\hat{F} = \left\{ f \epsilon F : \begin{matrix} f(\cdot, t) \text{ is constant over each cell in the} \\ \text{grid for time } t, \; t = 1, \ldots, T \end{matrix} \right\},$$

$$\hat{F}^+ = \{ f \epsilon \hat{F} : f(y, t) \geq 0 \qquad \text{for } y \epsilon Y, \; t = 1, \ldots, T \}.$$

In Chapter VI we show that $P'_T[\psi, h]$, the Gateaux differential of P_T at ψ in the direction h, is given by

$$P'_T[\psi, h] = \sum_{t=1}^{T} \int_Y E_{yt}\left[b'\left(\sum_{s=1}^{T} \psi(X_s, s)\right)\right] p_t(y)\, h(y, t) dy \qquad \text{for } \psi \epsilon F^+, \; h \epsilon K(\psi). \qquad \text{(IV-8)}$$

Let $\rho(j)$ be the region in Y which comprises the j^{th} cell, and let

$$\hat{D}_T(\psi, j, t) = \int_{\rho(j)} E_{yt}\left[b'\left(\sum_{s=1}^{T} \psi(X_s, s)\right)\right] p_t(y)dy \qquad \text{for } j \in J,\ t = 1, \ldots, T. \tag{IV-9}$$

Henceforth we shall consider all $f \in \hat{F}$ to be functions of j and t for $j \in J$, $t = 1, \ldots, T$.

For $\psi \in \hat{F}^+$, let $\hat{K}(\psi)$ be the cone of directions h such that $\psi + \theta h \in \hat{F}^+$ for all sufficiently small nonnegative values of θ. For $\psi \in \hat{F}^+$ and $h \in \hat{K}(\psi)$, (IV-8) becomes

$$P'_T[\psi, h] = \sum_{t=1}^{T} \sum_{j \in J} \hat{D}_T(\psi, j, t)\, h(j, t). \tag{IV-10}$$

We now state the analogs of Theorems 1 and 2 of Chapter VI for the class of plans $\hat{\Psi}(m)$. Let $\mathscr{E}_T^+$ be the nonnegative orthant in Euclidean T space.

THEOREM 1. Suppose b is concave and has a bounded nonnegative derivative b'. Then ψ^* is T-optimal within $\hat{\Psi}(m)$ if and only if there exists $(\lambda(1), \ldots, \lambda(T)) \in \mathscr{E}_T^+$ such that

$$\begin{aligned} \hat{D}_T(\psi^*, j, t) &= \lambda(t) && \text{if } \psi^*(j, t) > 0, \\ &\le \lambda(t) && \text{if } \psi^*(j, t) = 0, \text{ for } j \in J,\ t = 1, \ldots, T. \end{aligned} \tag{IV-11}$$

Proof. The proofs of sufficiency and necessity are completely parallel to the ones given for the proof of Theorem 1 of Chapter VI provided one uses $\hat{D}_T(\psi^*, j, t)$ in place of $D_T(\psi^*, y, t)$ and (IV-10) for the computation of P'_T.

Let E_{jt} denote expectation conditioned on the target being in $\rho(j)$ at time t, and define

$$p_t(j) = \int_{\rho(j)} p_t(y)dy \qquad \text{for } j \in J,\ t = 1, \ldots, T.$$

Then for the case of an exponential detection function, one can show that

$$\hat{D}_T(\psi, j, t) = E_{jt}\left[\exp\left(-\sum_{s\neq t} W(X_s)\ \psi(X_s, s)\right)\right]\ p_t(j)\ W(j)e^{-W(j)\ \psi(j,t)}$$

for any search plan $\psi \in \hat{\Psi}(m)$, $j \in J$, and $t = 1, \ldots, T$. As above we think of the sweep width function W and the allocation functions $\psi \in \hat{F}$ as being functions of (j, t) for $j \in J$, $t = 1, \ldots, T$. So that if we write $W(X_s)$ or $\psi^*(X_s, s)$ we understand this to mean the value of W or $\psi^*(\cdot, s)$ for the cell in which X_s falls.

THEOREM 2. Suppose the detection function is exponential and W is bounded. Then ψ^* is T-optimal within $\hat{\Psi}(m)$, if and only if there exists $(\lambda(1), \ldots, \lambda(T)) \in \mathscr{E}_T^+$ such that for $j = 1, \ldots, J$, $t = 1, \ldots, T$,

$$\begin{aligned} E_{jt}\left[\exp\left(-\sum_{s\neq t} W(X_s)\ \psi^*(X_s, s)\right)\right]\ p_t(j)\ W(j)e^{-W(j)\psi^*(j,t)} &= \lambda(t) \text{ if } \psi^*(j,t) > 0\ , \\ &\leq \lambda(t) \text{ if } \psi^*(j,t) = 0. \end{aligned} \qquad \text{(IV-12)}$$

Proof. The proof follows that given for Theorem 2 of Chapter VI using the necessary conditions in Corollary 2.1.6 of reference [h].

For $l = 1, \ldots,$ let $\hat{\psi}_{lT}$ be the member of $\hat{\Psi}(m)$ identified with ψ_{lT} in the manner discussed at the beginning of this section.

THEOREM 3. The sequence $\{\hat{\psi}_{1T}, \hat{\psi}_{2T}, \ldots, \}$ has a convergent subsequence and the limit $\hat{\psi}^*$ of this convergent subsequence is T-optimal within $\hat{\Psi}(m)$ for the Monte Carlo sample of target paths. In addition,

$$\lim_{l\to\infty} P_T[\hat{\psi}_{lT}] = P_T[\hat{\psi}^*]. \qquad \text{(IV-13)}$$

Proof. Since each $\hat{\psi}_{lT}$ satisfies

$$\sum_{j=1}^{\bar{J}} A(j)\ \hat{\psi}_{lT}(j, t) = m(t) \qquad \text{for } t = 1, \ldots, T,$$

it follows that $\hat{\psi}_{l T}(j, t) \leq m(t)/A(j)$ for $t = 1, \ldots, T$, $j \in J$. Hence for each j and t there is a sequence $\{l_k\}_{k=1, 2, \ldots}$ such that $\lim_{k \to \infty} \hat{\psi}_{l_k T}(j, t)$ exists. Because the number of grid cells is countable, one can choose a Cantor diagonal sequence to obtain a subsequence $\{l_i\}_{i = 1, 2, \ldots}$ such that $\lim_{i \to \infty} \hat{\psi}_{l_i T}(j, t)$ exists for all j and t. Let

$$\hat{\psi}^*(j, t) = \lim_{i \to \infty} \hat{\psi}_{l_i T}(j, t).$$

Since each $\hat{\psi}_{l_i T} \in \hat{\Psi}(m)$, $\hat{\psi}^* \in \hat{\Psi}(m)$. By the Lebesgue dominated convergence theorem and the fact that $P_T[\hat{\psi}_{(l+1)T}] \geq P_T[\hat{\psi}_{l T}]$, for $l = 1, \ldots,$ equation (IV-13) must hold.

We now show that $\hat{\psi}^*$ is T-optimal within $\hat{\Psi}(m)$. For the n^{th} sample path define

$$w_n^* = w_n \exp\left(- \sum_{s=1}^{T} W(j_s^n)\, \hat{\psi}^*(j_s^n, s)\right).$$

Then w_n^* is the probability that the target follows the n^{th} sample path and is not detected using the plan $\hat{\psi}^*$. Define

$$\tilde{g}_{\hat{\psi}^*}(j, t) = \sum_{\{n : j_t^n = j\}} w_n^* \exp(W(j)\, \hat{\psi}^*(j, t)) \qquad \text{for } t = 1, \ldots, T,\ j = 1, \ldots, T.$$

Observe that

$$\tilde{g}_{\hat{\psi}^*}(j, t) = E_{jt}\left[\exp\left(- \sum_{s \neq t} W(X_s)\, \hat{\psi}^*(X_s, s)\right)\right] p_t(j),$$

provided we understand $\{X_t,\ t = 1, \ldots, T\}$ to be the Monte Carlo sample of target paths. Another way to view $\tilde{g}_{\hat{\psi}^*}(j, t)$ is as a Monte Carlo estimate of the expectation of the right-hand side of the above equation where $\{X_t,\ t = 1, \ldots, T\}$ is the underlying motion process from which our sample paths are drawn. We shall take the

former point of view for this proof.

Let $K(t) = \Sigma_{j=1}^{J} \tilde{g}_{\hat{\psi}*}(j,t)$ and $g_{\hat{\psi}*}(\cdot,t) = \tilde{g}_{\hat{\psi}*}(\cdot,t)/K(t)$. We claim that ψ^* satisfies conditions (IV-12) of Theorem 2, i.e., there is a vector $(\lambda(1), \ldots, \lambda(T))$ of non-negative numbers such that

$$g_{\hat{\psi}*}(j,t)\, W(j) \exp(-W(j)\, \hat{\psi}^*(j,t)) = \lambda(t) \qquad \text{if } \hat{\psi}^*(j,t) > 0,$$

$$\leq \lambda(t) \qquad \text{if } \hat{\psi}^*(j,t) = 0,\ j = 1, \ldots, J,\ t = 1, \ldots, T. \tag{IV-14}$$

We prove this claim by supposing that for some t, condition (IV-14) does not hold. That is $\hat{\psi}^*(\cdot,t)$ fails to satisfy the necessary conditions of Corollary 2.1.6 of reference [h] for $\hat{\psi}^*(\cdot,t)$ to maximize probability of detection for cost m(t) for the stationary target problem with target probability distribution $g_{\hat{\psi}*}(\cdot,t)$ and exponential detection function with sweep width W(j) for the j^{th} cell. Thus one can find an allocation f* such that

$$\sum_{j=1}^{J} A(j)\, f^*(j) = m(t),$$

and

$$\sum_{j=1}^{J} g_{\hat{\psi}*}(j,t) \left[1 - \exp\left(-W(j)\, f^*(j)\right)\right] > \sum_{j=1}^{J} g_{\hat{\psi}*}(j,t) \left[1 - \exp\left(-W(j)\, \hat{\psi}^*(j,t)\right)\right].$$

This implies that

$$1 - P_T[\hat{\psi}^*] = \sum_{j=1}^{J} \tilde{g}_{\hat{\psi}*}(j,t) \exp(-W(j)\, \psi^*(j,t)) > \sum_{j=1}^{J} \tilde{g}_{\hat{\psi}*}(j,t) \exp(-W(j)\, f^*(j)). \tag{IV-15}$$

Let Ξ indicate the operation which maps a plan ψ into the plan $\Xi(\psi)$ which results from performing one pass of the algorithm. Then the nondecreasing nature of the

detection probability resulting from each of the stages forming a single pass combined with (IV-15) implies

$$P_T[\Xi(\hat{\psi}^*)] > P_T[\hat{\psi}^*]. \tag{IV-16}$$

Clearly Ξ is a continuous operator in the supremum norm so that

$$\lim_{i \to \infty} \Xi[\hat{\psi}_{l_i T}] = \Xi[\hat{\psi}^*].$$

However, since P_T is continuous and (IV-13) holds,

$$\begin{aligned} P_T[\Xi(\hat{\psi}^*)] &= \lim_{i \to \infty} P_T[\Xi(\hat{\psi}_{l_i T})] \\ &= \lim_{i \to \infty} P_T[\hat{\psi}_{(l_i+1)T}] \\ &= P_T[\hat{\psi}^*], \end{aligned}$$

which contradicts (IV-16). Thus we have shown that $\hat{\psi}^*$ satisfies conditions (IV-12) and by Theorem 2, $\hat{\psi}^*$ is T-optimal within $\hat{\Psi}(m)$ for the Monte Carlo sample of target paths.

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CHAPTER V

ALGORITHMS FOR OTHER SEARCH PROBLEMS

In this chapter we outline algorithms which can be used to solve a number of search problems related to the main one considered in this report. These algorithms are based on necessary and sufficient conditions for optimality which are related to the ones obtained in Chapter IV.

In the first section we outline an algorithm which can be used to calculate plans that maximize probability of detecting the target by time T when the detection function is not exponential. A method of calculating plans which minimize mean time to complete a search for a moving target is presented in the second section. The third section outlines an algorithm for finding optimal survivor search plans. The final section gives a method for maximizing probability of detection when there is no constraint on the rate at which effort may be applied but only on the total effort available for time 1 to T.

None of the algorithms in this chapter have been programmed or tested. Thus, the algorithms should be viewed as approaches to solving the problems and not necessarily as answers.

Optimal Plans for Non-Exponential Detection Functions

In this section, we consider the basic search problem stated in the first section of Chapter I when the exponential detection function is replaced by a more general

regular detection function b. A regular detection function $b : [0, \infty) \to [0, 1]$ has a positive, continuous, and strictly decreasing derivative b'. As in the third section of Chapter IV, we consider ψ to be a search density on the plane so that the probability of detection P_T in (I-3) of the problem statement in Chapter I becomes

$$P_T[\psi] = E\left[b\left(\sum_{t=1}^{T} \psi(X_t, t)\right)\right]. \tag{V-1}$$

As in Chapter IV we shall restrict ourselves to allocations which are constant over each cell of our search grid. That is, we consider only plans in the class $\hat{\Psi}(m)$ defined in Chapter IV. Recall that this restriction is not equivalent to assuming that the target is moving in discrete space. The target motion process $X = \{X_t, t=1, \ldots, T\}$ may take place in the discrete space composed of the grid cells or it may take place in the underlying space, the plane, over which the grid has been imposed. In either case

$$p_t(j) = \Pr\{X_t \text{ is in cell } j\} \qquad \text{for } j \in J,\ t = 1, \ldots, T.$$

As in Chapter IV, let E_{jt} denote expectation conditioned on the target being in cell j at time T. The necessary and sufficient conditions of Theorem 1 of Chapter VI for a plan $\psi^* \in \hat{\Psi}(m)$ to be T-optimal within $\hat{\Psi}(m)$ become

$$E_{jt}\left[b'\left(\sum_{s=1}^{T} \psi(X_s, s)\right)\right] p_t(j) \quad = \lambda(t) \quad \text{if } \psi^*(j,t) > 0$$

$$\leq \lambda(t) \quad \text{if } \psi^*(j,t) < 0,\ j \in J,\ t = 1, \ldots, T,$$

for some vector $(\lambda(1), \ldots, \lambda(T))$ of nonnegative numbers.

Description of the algorithm. As in Chapter IV we assume that the target motion process $\{X_t, t = 1, \ldots, T\}$ is approximated by a large but finite number of sample paths drawn from this process in a Monte Carlo fashion. That is, we suppose that we are given N sample paths drawn from the process $\{X_t; t=1, \ldots, T\}$. The n^{th} sample path is assumed to have the form

$$(x_1^n, x_2^n, \ldots, x_T^n),\ w_n,$$

where x_s^n is the position of the target at time s in the n^{th} sample path and w_n is the sample probability that the n^{th} sample path represents the target's actual path (usually $w_n = 1/N$).

Conceptually the first act of the optimizer is to convert the sample paths into sequences of cell numbers. Thus the n^{th} sample path becomes

$$(j_1^n, j_2^n, \ldots, j_T^n),\ w_n,$$

where j_s^n is the index of the cell into which x_s^n falls.

Let

$$\begin{aligned} z_n &= \text{total effort density that accumulates on path n} \\ &= \sum_{t=1}^{T} \psi(j^n, t) \qquad \text{for } n = 1, \ldots, N. \end{aligned} \tag{V-}$$

Then

$$E_{jt}\left[b'\left(\sum_{t=1}^{T} \psi(X_t, t)\right)\right] p_t(j) = \sum_{\{n: j_t^n = j\}} w_n b'(z_n). \tag{V-}$$

The algorithm proceeds as follows:

1. Let $\epsilon > 0$.

2. Let ψ_0 be an initial allocation and compute z_n for $n = 1, \ldots, N$.

3. Set $l = 0$.

4. Set $t = 1$.

5. Find the allocation $f: J \to [0, \infty]$ and nonnegative number $\lambda(t)$ such that

   $\Sigma_{j \in J} f(j)\, A(j) = m(t)$ for $j \in J$ and

$$\Sigma_{\{n: j_t^n = j\}} w_n b'\left(z_n - \psi_{lT+t-1}(j, t) + f(j)\right) \begin{array}{ll} = \lambda(t) & \text{if } f(j) > 0 \\ \leq \lambda(t) & \text{if } f(j) = 0. \end{array} \qquad \text{(V-5)}$$

6. Set

$$\psi_{lT+t}(j, s) = \begin{cases} \psi_{lT+t-1}(j, s) & \text{for } s \neq t \\ f(j) & \text{for } s = t \end{cases} \quad \text{for } j \in J.$$

7. Set $t = t+1$.

8. If $t \leq T$, go to step 5.

9. If $t > T$, set $l = l+1$ and compute $d = \left| P_T[\psi_{lT}] - P_T[\psi_{(l-1)T}] \right|$.

10. If $d \leq \epsilon$ stop; ψ_{lT} is the answer.

11. If $d > \epsilon$, set $t = 1$ and go to step 5.

The difficult part of this algorithm involves finding f and $\lambda(t)$ in step 5. One way to proceed is to choose a value for $\lambda(t)$ and then solve for $f(j)$ for $j \in J$ in (V-5). Since b' is monotone decreasing, this is a straightforward numerical problem. One then computes $\Sigma_{j \in J} f(j)$. If this sum is larger than m(t), then one should increase the value of $\lambda(t)$. Correspondingly if the sum is less than m(t), $\lambda(t)$ should be lowered.

Since the sum is a monotone function of $\lambda(t)$, a straightforward numerical search procedure can be used to find the value of $\lambda(t)$ such that $\Sigma_{j \epsilon J} f(j) = m(t)$.

The disadvantage of the above algorithm is that it requires a much larger number of calculations than the algorithm in Chapter IV for an exponential detection function.

Minimizing Mean Time to Complete a Search

In this section we outline an algorithm for computing plans which minimize mean time to complete a search when the target moves according to a discrete time and space finite Markov chain and the detection function is exponential as in Chapter II.

The search has a cutoff time T so that the search will proceed until the target is detected or time T is reached. Thus, if the target is not found by the end of time T, the search will stop. We say the search is completed if either the target is found or the search has been stopped after time T. The object is to minimize the mean time to complete the search.

Let $\mu_T[\psi]$ be the mean time to complete the search using plan ψ. For the remainder of this chapter, we consider ψ to be an allocation of effort as described in Chapter I. Define $P_0[\psi] = 0$ and let $P_t[\psi]$ be the probability of detection by time t using plan ψ. Then

$$\begin{aligned}\mu_T[\psi] &= \sum_{t=0}^{T} (1-P_t[\psi]) \\ &= 1 + \sum_{t=1}^{T} E\left[\exp\left(-\sum_{s=1}^{t} W(X_s)\psi(X_s, s)/A(X_s)\right)\right].\end{aligned}$$

Our problem is to find $\psi^* \epsilon \Psi(m)$ such that

$$\mu_T[\psi^*] = \min\{\mu_T[\psi] : \psi \epsilon \Psi(m)\}.$$

Following the argument in Chapter VI, one may calculate that the Gateaux differential of μ_T at ψ in the direction h is

$$\mu'_T[\psi, h] = -\sum_{u=1}^{T} \sum_{j\in J} E_{ju}\left[\sum_{t=u}^{T} \exp\left(-\sum_{s=1}^{t} W(X_s)\psi(X_s, s)/A(X_s)\right)\right] p_u(j)W(j)h(j,u)/A(j).$$

Thus a necessary and sufficient condition for $\psi^* \in \Psi(m)$ to satisfy (V-6) is that there exist nonnegative numbers $\lambda(t)$ for $t = 1, \ldots, T$ such that

$$\sum_{t=u}^{T} E_{ju}\left[\exp\left(-\sum_{s=1}^{t} W(X_s)\psi^*(X_s, s)/A(X_s)\right)\right] p_u(j)\frac{W(j)}{A(j)} \begin{array}{ll} = \lambda(u) & \text{if } \psi^*(j,u) > 0, \\ \le \lambda(u) & \text{if } \psi^*(j,u) = 0. \end{array} \tag{V-7}$$

For $j \in J$, $t = 1, \ldots, T$, and search plans ψ, define

$$\Delta_{ju}(t,\psi) = E_{ju}\left[\exp\left(-\sum_{\substack{s=1\\ s\neq u}}^{t} W(X_s)\psi(X_s, s)/A(X_s)\right)\right] p_u(j) \quad \text{for } u \le t.$$

Equation (V-7) may be written as

$$\left[\sum_{t=u}^{T} \Delta_{ju}(t,\psi^*)\right]\frac{W(j)}{A(j)} \exp(-W(j)\psi^*(j,u)/A(j)) \begin{array}{ll} = \lambda(u) & \text{for } \psi^*(j,u) > 0, \\ \le \lambda(u) & \text{for } \psi^*(j,u) = 0. \end{array}$$

Define R and τ as in the third section of Chapter II. Recall that

$R(j, t, \psi)$ = Probability the target reaches cell j at time t and is not detected by the effort at times $s = 1, \ldots, t-1$ under plan ψ,

and that τ is the transition function for the process. Assume that ρ_1 gives the initial distribution of the process at time 1 and that $\rho_t(j) = 1$ for all $t > 1$ and $j \in J$. Define

$V(j, t, s, \psi)$ = Probability that given the target starts in cell j at time t it will not be detected by the effort at times $u = t+1, \ldots, s$, under plan ψ.

It follows that for any plan ψ, $j \in J$, $u = 1, \ldots, T$, and $t \geq u$, that

$$\Delta_{ju}(t, \psi) = R(j, u, \psi)\, V(j, u, t, \psi).$$

We now describe the algorithm:

(1) (a) Set $l = 0$, $\psi_0(j, t) = 0$ for $j \in J$, $t = 1, \ldots, T$,

and $\mu_T[\psi_0] = T$.

(b) Choose $\epsilon > 0$ and compute $R(\cdot, \cdot, \psi_0)$.

(2) Set $u = T$.

(3) Compute

$$\Delta_{ju}(t, \psi_l) = R(j, u, \psi_l)\;\; V(j, u, t, \psi_l) \qquad \text{for } j \in J,\ t = u, \ldots, T.$$

(4) Find f: $\{1, \ldots, J\} \rightarrow [0, \infty]$ such that $\sum_{j \in J} f(j) = m(u)$ and

$$\left[\sum_{t=u}^{T} \Delta_{ju}(t, \psi_l)\right] \frac{W(j)}{A(j)}\, e^{-W(j)f(j)/A(j)} \begin{array}{ll} = \lambda(u) & \text{for } f(j) > 0, \\ \leq \lambda(u) & \text{for } f(j) = 0. \end{array}$$

(5) Set

$$\psi_{l+1}(j, s) = \begin{cases} \psi_l(j, s) & \text{for } s \neq u, \\ f(j) & \text{for } s = u. \end{cases}$$

(6) If $u = 1$, set $l = l+1$ and go to step (9).

(7) Compute

$$V(j, u-1, t, \psi_{l+1}) = \sum_{k \in J} \tau_{u-1}(j, k) \exp\left(-W(k)\; \psi_{l+1}(k, u)/A(j)\right) V(k, u, t, \psi_l)$$

for $j \in J$, $t = u, \ldots, T$.

(8) Set $u = u-1$, $l = l+1$ and go to step (3).

(9) Compute $R(\cdot, \cdot, \psi_l)$ and $\mu_T[\psi_l] = 1 + \sum_{t=1}^{T} \sum_{j \in J} R(j, t, \psi_l)\, e^{-W(j)\psi_l(j,t)/A(j)}$.

(10) Compute $d = |\mu_T[\psi_l] - \mu_T[\psi_{l-T}]|$.

(11) If $d > \epsilon$, go to step (2).

(12) If $d \leq \epsilon$, stop; ψ_l is the answer.

Note that step (4) is equivalent to finding the optimal allocation of m(t) effort for a density function d given by

$$d(j) = \sum_{t=u}^{T} \Delta_{ju}(t, \psi_l) \qquad \text{for } j \in J.$$

The algorithm in Example 2.2.8 of reference [h] will find such an f*.

Optimal Allocation for the Survivor Search Problem

In this section we consider a discrete time version of the survivor search problem addressed in reference [t]. The target is stationary with distribution function p where

$$p(j) = \Pr\{\text{target in cell } j\} \qquad \text{for } j \in J.$$

The target has a stochastic lifetime whose distribution may depend on the target's location. In particular, we assume that

$$\varphi(j,t) = \Pr\{\text{target dies at the end of time } t \mid \text{target is in cell } j\}$$
$$\text{for } j \in J,\ t = 0, 1, \ldots .$$

Note that $\Sigma_{j\in J}\,\varphi(j,0)$ is the probability that the target dies before the search begins. We shall assume that the target's lifetime is bounded by some time T so that

$$\sum_{t=0}^{T} \varphi(j,t) = 1 \quad \text{for } j \in J.$$

Let ψ be a search plan. Then, the probability $S_T[\psi]$ of finding the target alive with plan ψ is

$$S_T[\psi] = \sum_{j\in J} p(j) \sum_{t=0}^{T} \varphi(j,t) \left[1-\exp\left(-\frac{W(j)}{A(j)}\sum_{u=1}^{t} \psi(j,u)\right)\right].$$

Note, we follow the convention that $\sum_{u=1}^{0} = 0$. The survivor search problem is to find $\psi^* \in \Psi(m)$ such that

$$S_T[\psi^*] = \max\{S_T[\psi]:\psi \in \Psi(m)\}. \qquad \text{(V-}$$

Following the argument in reference [t] one can show that a necessary and sufficient condition for $\psi^* \in \Psi(m)$ to be an optimal survivor search plan is that there exist nonnegative numbers $\lambda(t)$ for $t = 1, \ldots, T$ such that

$$p(j) \sum_{t=s}^{T} \varphi(j,t) \frac{W(j)}{A(j)} \exp\left(-\frac{W(j)}{A(j)} \sum_{u=1}^{t} \psi^*(j,u)\right) \begin{cases} = \lambda(s) & \text{if } \psi^*(j,u) > 0, \\ \le \lambda(s) & \text{if } \psi^*(j,u) = 0. \end{cases} \qquad \text{(V-}$$

For any search plan ψ define

$$U(j,t,\psi) = \exp\left(-\frac{W(j)}{A(j)} \sum_{u=1}^{t} \psi(j,u)\right) \quad \text{for } j \in J,\ t = 1, \ldots, T.$$

Now, we can write (V-10) as

$$\frac{p(j)W(j)}{A(j)} \sum_{t=s}^{T} \varphi(j,t)\, U(j,t,\psi^*) \quad \begin{aligned} &= \lambda(s) && \text{if } \psi^*(j,s) > 0, \\ &\leq \lambda(s) && \text{if } \psi^*(j,s) = 0. \end{aligned} \tag{V-11}$$

For any plan ψ, define

$$\begin{aligned} \tilde{U}_s(j,t,\psi) &= \exp\left(-\frac{W(j)}{A(j)} \sum_{\substack{u=1 \\ u \neq s}}^{t} \psi(j,u)\right) \\ &= U(j,t,\psi) \exp\left(\frac{W(j)}{A(j)}\, \psi(j,s)\right) \qquad \text{for } j \in J,\ s \leq t \leq T. \end{aligned}$$

Then conditions (V-11) can be written as

$$\frac{p(j)W(j)}{A(j)} \left[\sum_{t=s}^{T} \varphi(j,t)\, \tilde{U}_s(j,t,\psi^*)\right] \exp\left(-\frac{W(j)}{A(j)}\, \psi^*(j,s)\right) \quad \begin{aligned} &= \lambda(s) && \text{if } \psi^*(j,s) > 0, \\ &\leq \lambda(s) && \text{if } \psi^*(j,s) = 0. \end{aligned}$$

The following gives a description of an algorithm for calculating optimal survivor search plans:

(1) Choose $\epsilon > 0$ and set $l = 0$.

(2) Set $\psi_0(j,t) = 0$ for $j \in J$, $t = 1, \ldots, T$.

(3) Set $U(j,t,\psi_0) = 1$ for $j \in J$, $t = 1, \ldots, T$.

(4) Set $s = T$.

(5) Compute

$$\tilde{U}_s(j,t,\psi_l) = U(j,t,\psi_l) \exp\left(\frac{W(j)}{A(j)}\, \psi_l(j,s)\right) \qquad \text{for } j \in J,\ t = s, \ldots, T.$$

(6) Solve for $f : J \to [0,\infty)$ such that $\Sigma_{j\in J} f(j) = m(t)$ and

$$\frac{p(j)W(j)}{A(j)}\left[\sum_{t=s}^{T} \varphi(j,t)\,\tilde{U}(j,t,\psi_l)\right]\exp\left(-\frac{W(j)}{A(j)}f(j)\right) \begin{array}{l} = \lambda(s) \text{ if } f(j) > 0, \\ \leq \lambda(s) \text{ if } f(j) = 0. \end{array}$$

(7) Set

$$\psi_{l+1}(j,t) = \begin{cases} \psi_l(j,t) & \text{for } t \neq s \\ f(j) & \text{for } t = s \end{cases} \quad \text{for } j \in J.$$

(8) If $s = 1$, go to step (11), otherwise got to step (9).

(9) Compute

$$U(j,t,\psi_{l+1}) = \tilde{U}_s(j,t,\psi_l)\exp\left(-\frac{W(j)}{A(j)}\psi_{l+1}(j,s)\right) \quad \text{for } j\in J,\ t = s, \ldots, T.$$

(10) Set $s = s-1$, $l = l+1$, and go to step (5).

(11) Set $l = l+1$ and compute $d = \left|S_T[\psi_l] - S_T[\psi_{l-T}]\right|$.

(12) If $d > \epsilon$ go to step (4).

(13) If $d \leq \epsilon$ stop; ψ_l is the answer.

Observe that $\Sigma_{j\in J}\, p(j)\left[\Sigma_{t=s}^{T}\varphi(j,t)\,U(j,t,\psi)\right] = \Sigma_{t=s}^{T}\Sigma_{j\in J}\, p(j)\,\varphi(j,t)U(j,t,\psi)$ is the probability that the target dies in the interval $[s,T]$ before being detected by plan ψ. Step (6) of the algorithm finds the allocation of effort for time s which minimizes this probability given that the allocation at all times other than s is already specified. Observe that for any plan ψ

$$1 - S_T[\psi] = \sum_{t=0}^{T}\sum_{j\in J} p(j)\,\varphi(j,t)\,U(j,t,\psi). \qquad \text{(V-1}$$

The optimization procedure first reallocates effort at time T to minimize the t = T term in the above sum. Then it reallocates the effort at T-1 to minimize the sum of the t = T-1 and T terms. For time s it reallocates the effort for that time to minimize the sum of the t = s to t = T terms in (V-12). Thus, $S_T[\psi_{l+1}] \geq S_T[\psi_l]$ for $l = 0, 1, \ldots$.

Notice that finding the solution f in step (6) is equivalent to finding the allocation of m(s) effort which maximizes the probability of detecting a stationary target with defective location distribution δ given by

$$\delta(j) = p(j) \sum_{t=s}^{T} \varphi(j,t)\, \bar{U}(j,t,\psi_l) \text{ for } j \in J,$$

and exponential detection function. This allocation may be performed by the algorithm given in Example 2.2.8 of reference [h].

One can show that the above algorithm converges by using an argument similar to the one given in the third section of Chapter III.

Allocating the Total Effort Available

For the other search problems considered in this report we have assumed that search effort becomes available at the rate m(t) for t = 1, ..., T. By contrast, in this section we assume that there is a total amount M of effort which can be applied at any rate the planner wishes. With the exception of having the constraint on total effort rather than on the rate at which effort can be applied, we return to the basic search problem described in Chapter I.

This type of problem occurs when there is only one aircraft flight available to search for a submarine in some interval t = 1, ..., T of time and the search planner wishes to select the best time to search. Solving this problem is equivalent to solving T

stationary target problems. One simply computes the target's distribution at time t given that no search has been applied and finds the optimal allocation of M units of effort to that distribution for $t = 1, \ldots, T$. The planner then chooses the time t^* which yields the highest probability of detection and follows the optimal allocation for that time.

A more interesting problem arises when one is not required to allocate all of his effort during a single time period. In this case one can show that a necessary and sufficient condition for a plan ψ^*, such that $\Sigma_{j\in J} \Sigma_{t=1}^{T} \psi^*(j,t) = M$, to maximize the probability of detection by time T within the class of plans which allocate M units of effort is the existence of a $\lambda \geq 0$ such that for $j \in J$, $t = 1, \ldots, T$

$$E_{jt}\left[\exp\left(-\frac{W(j)}{A(j)} \sum_{s\neq t} \psi^*(X_s, s)\right)\right] p_t(j) \frac{W(j)}{A(j)} \exp\left(-\frac{W(j)}{A(j)} \psi^*(j,t)\right) \begin{array}{l} = \lambda \text{ if } \psi^*(j,t) > 0, \\ \leq \lambda \text{ if } \psi^*(j,t) = 0. \end{array} \tag{V-13}$$

For a search plan ψ let

$$g_\psi(j,t) = E_{jt}\left[\exp\left(-\frac{W(j)}{A(j)} \sum_{s\neq t} \psi(X_s, s)\right)\right] p_t(j) \quad \text{for } t = 1, \ldots, T, \text{ and } j \in J.$$

The following is a proposed algorithm for solving the above problem:

(1) Make an initial guess ψ_0 for the search plan and choose $\epsilon_1, \epsilon_2 > 0$.

(2) Choose $\lambda_0 > 0$ and set $l = 0$ and $k = 0$.

(3) For $t = 1, \ldots, T$ do steps (4) - (6).

(4) Compute $g_{\psi_l}(\cdot, t)$ and solve the following equation for $f(j)$:

$$g_{\psi_l}(j,t) \frac{W(j)}{A(j)} \exp\left(-\frac{W(j)}{A(j)} f(j)\right) = \lambda_k \quad \text{for } j \in J.$$

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(5) Let $f^+(j) = \max\{0, f(j)\}$ for $j \in J$ and set

$$\psi_{l+1}(j, s) = \begin{cases} \psi_l(j, s) & \text{for } s \neq t \\ f^+(j) & \text{for } s = t. \end{cases}$$

(6) Set $l = l + 1$.

(7) If $l > 0$, and $\max\limits_{j \in J,\ t=1,\ldots,T} \{ |\psi_l(j,t) - \psi_{l-T}(j,t)| \} \leq \epsilon_1$ go to step (9).

(8) Otherwise go to step (3).

(9) Calculate $C[\psi_l] = \sum\limits_{j \in J} \sum\limits_{t=1}^{T} \psi_l(j,t)$.

(10) If $|C[\psi_l] - M| < \epsilon_2$, stop; ψ_l is the answer.

(11) Otherwise set $k = k+1$ and choose λ_k to be larger than λ_{k-1} if

$C[\psi_l] > M$ and to be smaller than λ_{k-1} if $C[\psi_l] < M$.

In steps (3) - (8) the algorithm is recursively calculating an allocation ψ which satisfies (V-13) for $\lambda = \lambda_k$. Once this is done, at least approximately, the algorithm checks the total effort $C[\psi]$ associated with this allocation. If the total effort is too large (small), then λ_{k+1}, the next guess for λ, is made larger (smaller) than λ_k. Because $C[\psi]$ is a monotone decreasing function of λ, this will cause the $C[\psi]$ resulting from λ_{k+1} to be smaller (larger) than that obtained from λ_k. Thus steps (9) - (11) should really be thought of as performing a binary search to obtain a value of λ such that $C[\psi] = M$.

CHAPTER VI

GENERAL NECESSARY AND SUFFICIENT CONDITIONS FOR MOVING TARGET PROBLEMS

In this chapter* we find necessary and sufficient conditions for a search plan to maximize the probability of detecting a moving target by time T under constraints on the rate at which search effort may be applied. These conditions apply to a very wide variety of moving target problems in continuous or discrete time and continous or discrete space. Many previous results concerning necessary and sufficient conditions for moving target problems appear as special cases of the results obtained here. In particular our results include the necessary conditions obtained by Hellman, reference [u], for diffusion processes and by Saretsalo, reference [v], for continuous time and space Markov processes, and they extend those results by showing that the conditions are also sufficient. The results of Stone, reference [w], and Persiheimo, reference [x], for continuous time generalized conditionally deterministic motion are special cases of the results in this paper as well as those of Brown, reference [b], for discrete time and space target motion.

The results proved in this chapter are not restricted to problems in which targets move according to a Markov process or a mixture of Markov processes. The results apply to any process for which the expectation defining the function D_T in (VI-4) makes sense. In the case where the detection function is exponential, calculation of D_T is equivalent to being able to calculate, for each time t and point y in the search space,

* This chapter is based on reference [e].

the probability (density) that the target passes through point y at time t and is not detected by the search effort over time [0, T].

In the case of a discrete time target motion process and an exponential detection function, the necessary and sufficient conditions have the simple intuitive interpretation given in the basic condition of Chapter I. This condition forms the basis for the algorithms in Chapters II and IV.

In the first section of this chapter we state the generalized version of the optimal search problem of Chapter I that we are considering here. The statement of the problem and the theorems are given in terms of a discrete time and continuous space target motion model. Modifications required to apply the results to discrete space or continuous time are noted after the results are stated. A discrete time model is used because it is most amenable to numerical calculation and because discrete time allows us to present proofs that are considerably simpler than those required for continuous time. A continuous space is chosen for the basic presentation because it illustrates that the results are not simply applications of the Kuhn-Tucker theorem. In addition, the discrete-space results are usually transparent once the continuous-space results have been obtained.

In Theorem 1 of the second section we prove that conditions (IV-6), stated below, are necessary and sufficient for the optimality of a search plan in discrete time and continuous space when the detection function is concave. In the case of continuous time, we observe that the necessary conditions (VI-6) are true but their proof is not a simple extension of the one given in Theorem 1. Instead the reader is referred to the proof of Theorem 5.2 of reference [w] which may easily be applied to proving this result. However, the proof in reference [w] is very technical involving a demonstration of the

existence of a measurable selection from a function space. This difficulty is not present in discrete time. In Theorem 1' of the second section we present a unified statement of the necessary and sufficient conditions which applies to a target motion process with any combination of discrete or continuous space and time.

The third section considers the special case of a discrete time target motion process and an exponential detection function. For this case we give a simple proof of the necessity result which relies only on the necessary conditions for an optimal search plan for a stationary target. In this case, the necessary and sufficient conditions have the simple intuitive interpretation given in the basic condition of Chapter I; namely, at each time t, the optimal plan allocates its effort for that time period so as to maximize the probability of detecting a stationary target with location distribution equal to the posterior target location distribution given failure to detect at all times other than t, i.e., at all times before <u>and</u> after time t. For discrete time and space target motion, Washburn, reference [y], has shown that the basic condition is necessary but not sufficient when the searcher obtains independent glimpses at the target at each time period, but the detection function during a single time period is not exponential.

In order to avoid confusion, the reader should note that we have adopted a different detection model from that used by Washburn. In the discussion in the first section below, the reader will see that we have assumed a constant detection function b over all time periods and that this function relates the total effort which falls on the target during the search to the probability of detecting the target. Only when b is an exponential detection function does one have independent glimpses at the target at each time period. Thus, the model assumed by Washburn coincides with the one used in this report only when the detection function is exponential.

Problem Statement

Let*$\{X_t; t = 0, 1, \ldots, T\}$ be a discrete time stochastic process where X_t takes values in Euclidean n-space, Y, and T is a positive integer. The random variable X_t represents the target's position at time t. We assume that $(X_0, X_1, \ldots, X_T)$ has a joint density function p defined on Y^{T+1}. Let p_t be the density of the marginal distribution of X_t for $t = 0, 1, \ldots, T$.

Let $\psi(\cdot, t)$ be the allocation of search density at time t for $t = 0, 1, \ldots, T$. That is, $\psi(y, t)$ is the effort density applied to point y at time t under plan ψ. We assume that ψ is a member of the space F of real-valued Borel functions f defined on $Y \times \{0, 1, \ldots, T\}$ such that

$$\int_Y |f(y,t)|\,dy < \infty \qquad \text{for } t = 0, 1, \ldots, T,$$
$$||f|| \equiv \sum_{t=0}^{T} \text{ess sup}\, |f(y,t)| < \infty.$$

Then F is a linear space with norm $||f||$ for $f \in F$. Let $F^+ = \{f \in F : f \geq 0\}$, and let $m(t) > 0$ be the amount of effort which is available at time t for $t = 0, 1, \ldots, T$. Define

$$\Psi(m) = \{\psi \in F^+ : \int_Y \psi(y,t)\,dy = m(t) \text{ for } t = 0, 1, 2, \ldots, T\}. \qquad \text{(VI-1)}$$

Let $X(\omega, \cdot)$ denote a sample path of the process X. If the target follows this path and we allocate search effort according to the plan ψ, then

$$b\left(\sum_{s=0}^{T} \psi(X(\omega, s), s)\right)$$

is the probability of detecting the target by time T. The function b is called a detection function. It relates the accumulated search density along the target's path to probability

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* In distinction to previous chapters, we take the initial time to be 0. This is to facilitate a combined statement of continuous and discrete time results in Theorem 1'.

of detection. Letting E indicate expectation over sample paths, we have that the overall probability of detection by time T is

$$P_T[\psi] \equiv E\left[b\left(\sum_{s=0}^{T} \psi(X(\omega, s), s) \right) \right].$$

In the sequel we shall usually not indicate the dependence of X on ω.

The optimal search problem under consideration is to find $\psi^* \in \Psi(m)$ such that

$$P_T[\psi^*] = \max\{ P_T[\psi] \epsilon \Psi(m)\}. \qquad \text{(VI-}$$

A plan $\psi^* \epsilon \Psi(m)$ that satisfies (VI-2) is called T-optimal within $\Psi(m)$.

Necessary and Sufficient Conditions

In this section we find necessary and sufficient conditions for a plan ψ^* to be T-optimal within $\Psi(m)$ when the detection function b is concave. The conditions involve the Gateaux differential of P_T which we will now define and calculate under the assumption that for some finite $\kappa > 0$, the derivative b' of b satisfies $0 \leq b'(z) \leq \kappa$ for $z \geq 0$.

Gateaux differential of P_T. Let $\psi, h \epsilon F$. If

$$P'_T[\psi, h] \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(P_T[\psi + \epsilon h] - P_T[\psi] \right)$$

exists, then $P'_T[\psi, h]$ is the Gateaux differential of P_T at ψ in the direction h.

For $\psi \epsilon F^+$, let $C(\psi)$ be the cone of directions h such that $\psi + \theta h \epsilon F^+$ for all sufficiently small nonnegative values of θ. Now for $\psi \epsilon F^+$ and $h \epsilon C(\psi)$,

$$P'_T[\psi, h] = \lim_{\epsilon \to 0} E\left\{ \frac{1}{\epsilon} \left[b\left(\sum_{s=0}^{T} \psi(X_s, s) + \epsilon h(X_s, s) \right) - b\left(\sum_{s=0}^{T} \psi(X_s, s) \right) \right] \right\}.$$

Since the integrad is bounded by $\kappa \| h \|$, we may invoke the dominated convergence theorem to obtain

$$P'_T[\psi, h] = E\left\{ b'\left(\sum_{s=0}^{T} \psi(X_s, s) \right) \sum_{t=0}^{T} h(X_t, t) \right\}$$

$$= E\left\{ \sum_{t=0}^{T} b'\left(\sum_{s=0}^{T} \psi(X_s, s) \right) h(X_t, t) \right\}. \tag{VI-3}$$

Let p_t denote the marginal density for X_t, and let E_{yt} denote expectation conditioned on $X_t = y$. Then for $\psi \epsilon F^+$ and $h \epsilon C(\psi)$,

$$P'_T[\psi, h] = \sum_{t=0}^{T} \int_Y E_{yt} \left[b'\left(\sum_{s=0}^{T} \psi(X_s, s) \right) \right] h(y, t)\, p_t(y) dy.$$

Note that we have expressed $P'_T[\psi, \cdot]$ as a linear functional on $C(\psi)$. Define

$$D_T(\psi, y, t) = E_{yt} \left[b'\left(\sum_{s=0}^{T} \psi(X_s, s) \right) \right] \quad p_t(y) \text{ for } \psi \epsilon F^+,\ y \epsilon Y,\ t = 0, 1, \ldots, T. \tag{VI-4}$$

Then

$$P'_T[\psi, h] = \sum_{t=0}^{T} \int_Y D_T(\psi, y, t)\, h(y, t) dy \qquad \text{for } h \epsilon C(\psi). \tag{VI-5}$$

Necessary and sufficient conditions. We now state and prove necessary and sufficient conditions for T-optimality. Let $\mathscr{E}_{T+1}$ be Euclidean T+1-space and $\mathscr{E}^+_{T+1}$ be the nonnegative orthant in $\mathscr{E}_{T+1}$.

THEOREM 1. *Suppose that b is concave and that it has a bounded nonnegative derivative,* b'. *Then* ψ^* *is T-optimal within* $\Psi(m)$ *if and only if there exist* $(\lambda(0), \ldots, \lambda(T)) \epsilon \mathscr{E}^+_{T+1}$ *such that*

$$D_T(\psi^*, y, t) = \lambda(t) \quad \text{if } \psi^*(y,t) > 0,$$
$$\leq \lambda(t) \quad \text{if } \psi^*(y,t) = 0 \quad \text{for a.e. } y \in Y,\ t = 0, 1, \ldots, T. \tag{VI-6}$$

Proof of sufficiency. The proof of sufficiency follows that of Theorem 8.4.1 in reference [h]. The essence of this argument is due to D. H. Wagner.

Observe that the concavity of b implies that P_T is a concave functional on F^+. We now proceed to use an argument by contradiction. Suppose that $\psi^* \epsilon \Psi(m)$ satisfies (VI-6) and that there is a $\psi \epsilon \Psi(m)$ such that $P_T[\psi] > P_T[\psi^*]$.

Since P_T is concave we have for $0 \leq \theta \leq 1$,

$$\begin{aligned} P_T[\psi^* + \theta(\psi - \psi^*)] - P_T[\psi^*] &= P_T[(1-\theta)\psi^* + \theta\psi] - P_T[\psi^*] \\ &\geq (1-\theta)P_T[\psi^*] + \theta P_T[\psi] - P_T[\psi^*] \\ &= \theta(P_T[\psi] - P_T[\psi^*]). \end{aligned}$$

It follows that

$$P'_T[\psi^*, \psi - \psi^*] \geq P_T[\psi] - P_T[\psi^*] > 0. \tag{VI-7}$$

However, by equations (VI-5) and (VI-6) we have

$$\begin{aligned} P'[\psi^*, \psi - \psi^*] &= \sum_{t=0}^{T} \int_Y D_T(\psi^*, y, t)[\psi(y,t) - \psi^*(y,t)]dy \\ &= \sum_{t=0}^{T} \int_{\{y:\psi^*(y,t)>0\}} D_T(\psi^*, y, t)[\psi(y,t) - \psi^*(y,t)]dy \\ &\quad + \sum_{t=0}^{T} \int_{\{y:\psi^*(y,t)=0\}} D_T(\psi^*, y, t)[\psi(y,t) - \psi^*(y,t)]dy \\ &\leq \sum_{t=0}^{T} \int_Y \lambda(t)[\psi(y,t) - \psi^*(y,t)]dy = 0, \end{aligned} \tag{VI-8}$$

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where the last equality follows from the fact that

$$\int_Y \psi(y,t)dy = \int_Y \psi^*(y,t)dy = m(t) \quad \text{for } t = 0, \ldots, T.$$

However (VI-8) contradicts (VI-7) and sufficiency is proved.

Proof of necessity. Suppose ψ^* is optimal within $\Psi(m)$. Since b is increasing, we observe that ψ^* is also optimal within the larger class obtained by replacing the equality constraint by an inequality constraint in the definition of $\Psi(m)$ in (VI-1).

Since $\int_Y f(y,t)dy < \infty$ for $f \epsilon F^+$ and $t = 0, \ldots, T$, we may define

$$L_\gamma[f] = P_T[f] - \sum_{t=0}^{T} \gamma(t) \left[\int_Y f(y,t)dy - m(t)\right] \quad \text{for } f \epsilon F^+,\ \gamma \epsilon \mathscr{E}_{T+1}.$$

By Theorem 1 on page 217 of reference [z], there exists $\lambda \epsilon \mathscr{E}^+_{T+1}$ such that $L_\lambda[\psi^*] = \max_{\psi \epsilon F^+} L_\lambda[\psi]$.

For $f \epsilon F^+$ and $h \epsilon C(f)$, the Gateaux differential $L'_\lambda[f,h]$ of L_λ at f in the direction of h exists and is a linear functional of h. In particular by (VI-5)

$$L'_\lambda[f,h] = \sum_{t=0}^{T} \int_Y D_T(f,y,t)\, h(y,t)dy - \sum_{t=0}^{T} \lambda(t) \int_Y h(y,t)dy \qquad \text{(VI-9)}$$

Following page 227 of reference [z], we observe that since $L_\lambda[\psi^*] = \max_{\psi \epsilon F^+} L_\lambda[\psi]$, we have for any $\psi \epsilon F^+$,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \{L_\lambda[\psi^* + \epsilon(\psi - \psi^*)] - L_\lambda[\psi^*]\} \le 0.$$

Thus

$$L'_\lambda[\psi^*, \psi - \psi^*] \le 0 \quad \text{for } \psi \epsilon F^+. \qquad \text{(VI-10)}$$

Setting $\psi = \frac{1}{2}\psi^*$, we obtain

$$0 \geq L'_\lambda[\psi^*, -\tfrac{1}{2}\psi^*] = -\tfrac{1}{2} L'_\lambda[\psi^*, \psi^*], \tag{VI-11}$$

while setting $\psi = 2\psi^*$ yields

$$L'_\lambda[\psi^*, \psi^*] \leq 0. \tag{VI-12}$$

Equations (VI-9), (VI-11), and (VI-12) imply

$$\sum_{t=0}^{T} \int_Y [D_T(\psi^*, y, t) - \lambda(t)]\, \psi^*(y,t)dy = 0, \tag{VI-13}$$

while equations (VI-10) and (VI-13) imply

$$\sum_{t=0}^{T} \int_Y [D_T(\psi^*, y, t) - \lambda(t)]\, \psi(y,t)dy \leq 0 \qquad \text{for all } \psi \in L^+. \tag{VI-14}$$

Equation (VI-14) implies that $D_T(\psi^*, y, t) \leq \lambda(t)$ for a.e. $y \in Y$, $t = 0, 1, \ldots, T$, and (VI-13) implies $D_T(\psi^*, y, t) = \lambda(t)$ for a.e. $y \in Y$ and $t = 0, 1, \ldots, T$ such that $\psi^*(y,t) > 0$. The necessity of the conditions in (VI-6) follows, and the theorem is proved.

In the following paragraphs we discuss extensions and specializations of Theorem 1.

Discrete space - discrete time. Theorem 1 and the proof given above also hold when the search space Y is discrete provided one interprets $p_t(y)$ in (VI-4) as the probability that $X_t = y$.

Continuous time (sufficiency). The obvious analog of the sufficiency part of Theorem 1 holds for continuous time provided the target motion process $\{X_t;\ 0 \leq t \leq T\}$ has Borel measurable sample paths, the conditional expectation $E_{yt}[b'(\int_0^T \psi(X_s, s)ds]$ is well defined, and we take

$$D_T(\psi, y, t) = E_{yt}\left[b'\left(\int_0^T \psi(X_s, s)ds\right)\right] p_t(y) \text{ for } \psi \in F^+,\ y \in Y,\ t \in [0, T]. \tag{VI-15}$$

The proof of this assertion parallels the sufficiency proof in Theorem 1. The Borel measurability of the sample paths is needed to guarantee that the integral $\int_0^T \psi(X_s, s)ds$ is well defined.

Continuous time-continuous space (necessity). The corresponding necessity result for continuous time is more difficult to prove; its proof is not an obvious extension of the one given for Theorem 1. However, by paralleling the proof of Theorem 5.2 in reference [w] one may show that the conditions in (VI-6) are necessary with D_T defined as in (VI-15). In fact, that proof shows that for continuous space the necessary result holds when the concavity assumption on b is dropped. The proof in reference [w] also allows one to add a constraint of the form $0 \leq \psi(y, t) \leq B$ for some positive number B with a corresponding change in the conditions in (VI-6).

Observe that when $b(z) = 1 - e^{-z}$, we have, for continuous time,

$$D_T(\psi, y, t) = E_{yt}\left[\exp\left(-\int_0^T \psi(X_s, s)ds\right)\right] \; p_t(y). \tag{VI-16}$$

If $\{X_s; s \geq 0\}$ is a Markov process, then one can show that

$$D_T(\psi, y, t) = \int_X r(x, 0, y, t, \psi) \, R(y, t, T, \psi) \, p_0(x)dx, \tag{VI-17}$$

where under plan ψ, $r(x, 0, yt, \psi)$ is the probability density that at time t the target is located at y and is undetected given it was at x at time 0, and $R(y, t, T, \psi)$ is the probability that if the target is at point y at time t it will be undetected in the interval [t, T]. Because of the Markov nature of the process, $r(x, 0, y, t, \psi) \, R(y, t, T, \psi)$ is the probability density that a target starting at x at time 0 will pass through the point y at time t and remain undetected throughout [0, T]. The integral on the right of (VI-17) averages over the distribution of the target's position at time 0 to obtain the probability density that

the target passes through point y at time t and remains undetected throughout [0, T]. Now

$$\exp\left(-\int_0^T \psi(X(\omega, s), s)ds\right)$$

is the probability of failing to detect the target by time T given it follows path ω. Thus, the right-hand side of (VI-16) is simply the probability density of the target passing through point y at time t and failing to be detected by time T. From this observation and the above discussion, equation (VI-17) follows.

Thus, Saretsalo's Theorem 5.1 (reference [v]) and the theorem of Hellman (reference [u]) are special cases of the necessity result obtained in this chapter. In fact, the specialization to Markov processes given here is stronger than the result in reference [v] in the sense that no assumptions concerning the continuity of the transition function are required. In addition, we have proved sufficiency.

It also follows that the necessary and sufficient conditions obtained by Stone in reference [w] and Persiheimo in reference [x] are a special case of the conditions found in this chapter. In particular, one can show, in the notation of Stone (reference [w]) that

$$p_t(y) = \int_\Omega \frac{p(\eta_{\omega t}^{-1}(y), \omega)}{J(\eta_{\omega t}^{-1}(y), \omega, t)} \gamma(d\omega)$$

and

$$E_{yt}\left[b'\left(\int_0^T \psi(X_s, s)ds\right)\right] =$$

$$\frac{1}{p_t(y)} \int_\Omega \frac{p(\eta_{\omega t}^{-1}(y), \omega)}{J(\eta_{\omega t}^{-1}(y), \omega, t)} b'\left(\int_0^T \psi(\eta_{\omega s}[\eta_{\omega t}^{-1}(y)], s)ds\right) \gamma(d\omega),$$

so that $D_T(\psi, y, t) = E_{yt}[b'(\int_0^T \psi(X_s, s)ds)]\, p_t(y)$ coincides with the definition of D_T in Stone (reference [w]), and the conditions in that reference and in Persiheimo (reference [x]) are a special case of conditions (VI-6) provided one makes the obvious changes for the bound B on effort density which is allowed as a constraint by Stone reference [w].

Continuous time-discrete space (necessity). The necessity part of Theorem 1 also holds for a continuous time and discrete space motion process. Again the proof is not a simple extension of the one given for Theorem 1, but one can parallel the proof of Theorem 5.2 in reference [w] to obtain the result. However, in the case of discrete space, the assumption of concavity for b is required in order to guarantee the necessity of conditions (VI-6). The concavity is needed on page 464 of reference [u] where one invokes a Lagrange multiplier result to guarantee the existence of $\lambda(t)$ to satisfy (5.9) at the bottom of that page. In the case of a discrete search space one must invoke a result such as Corollary B.1.2 of reference [h] which requires the concavity of the detection function b.

Unified statement of results. Most of the above results can be consolidated into a single theorem statement provided we make the appropriate identifications for D_T, p_t, and $[0, T]$. Specifically, D_T is given by (VI-4) when time is discrete and by (VI-15) when time is continuous; p_t is the probability density function for X_t when Y is Euclidean n space, and $p_t(y) = \Pr\{X_t = y\}$ when Y is a discrete space. For continuous time F, F^*, $\psi(m)$, and P_T are defined as in the first section but with integrals replacing summation. In discrete time we understand $[0, T] = \{0, 1, \ldots, T\}$ while in continuous time $[0, T]$ has the usual meaning.

THEOREM 1'. Suppose b is concave and that it has a bounded nonnegative derivative b'. Assume that the sample paths of $\{X_t;\ 0 \leq t \leq T\}$ are Borel measurable and that D_T is well defined for (y, t) such that $p_t(y) > 0$. Then ψ^* is T-optimal within $\Psi(m)$ if and only if there exists $\lambda: [0, T] \rightarrow [0, \infty)$ such that

$$\begin{aligned} D_T(\psi^*, y, t) &= \lambda(t) \quad \text{if } \psi^*(y,t) > 0, \\ &\leq \lambda(t) \quad \text{if } \psi^*(y,t) = 0 \text{ for a.e. } (y,t) \in Y \times [0,T]. \end{aligned} \tag{VI-18}$$

In the case where the search space is $\mathscr{E}_n$, the necessity of conditions (VI-18) remains true when the concavity assumption on b is dropped. In discrete time, the sample paths will always be Borel measurable, and D_T will be well defined for (y, t) such that $p_t(y) > 0$.

The Special Case of an Exponential Detection Function

When the detection function is exponential, we may prove the necessity of the conditions in (VI-6) in an elementary manner which requires the use of only the necessary conditions for an optimal stationary target search. The use of the exponential detection function also allows us to consider the possibility that the detection capability of the search sensor varies over the search space. Specifically we shall assume that for each point $y \in Y$, there is a number W(y) which characterizes the detection performance of the sensor in the neighborhood of y in the sense that if the target is located at y and z effort density is placed there, then $1 - \exp(-W(y)z)$ is the probability of detecting the target. Classically W(y) is called the sweep width of the sensor when operating in the neighborhood of y (where 1/z has dimensions of distance).

For this case

$$P_T[\psi] = 1 - E\left[\exp\left(-\sum_{s=0}^{T} W(X_s)\,\psi(X_s, s)\right)\right] \quad \text{for } \psi \in F^+, \tag{VI-19}$$

and D_T, the kernel of the linear functional $P'_T[\psi, \cdot]$, becomes

$$D_T(\psi, y, t) = E_{yt}\left[\exp\left(-\sum_{s=0}^{T} W(X_s)\,\psi(X_s, s)\right)\right] p_t(y)\,W(y)$$

$$\text{for } \psi \in F,\ y \in Y,\ t = 0, 1, \ldots, T.$$

THEOREM 2. Suppose the detection function is exponential and W is bounded. Then ψ^* is T-optimal within $\Psi(m)$, if and only if there exists $(\lambda(0), \ldots, \lambda(T)) \in \mathscr{E}^+_{T+1}$ such that for $t = 0, \ldots, T$ and for a.e. $y \in Y$,

$$E_{yt}\left[\exp\left(-\sum_{s \neq t} W(X_s)\,\psi^*(X_s, s)\right)\right] p_t(y)\,W(y)e^{-W(y)\psi^*(y,t)} \begin{array}{l} = \lambda(t) \text{ if } \psi^*(y,t) > 0 \\ \leq \lambda(t) \text{ if } \psi^*(y,t) = 0. \end{array} \tag{VI-20}$$

Proof. Since P_T as defined in (VI-19) is a concave functional, the sufficiency proof for Theorem 1 applies here also.

To prove the necessity part of the theorem, let

$$\tilde{g}_t(y) = E_{yt}\left[\exp\left(-\sum_{s \neq t} W(X_s)\,\psi^*(X_s, s)\right)\right] p_t(y) \quad \text{for } y \in Y,\ t = 0, 1, \ldots, T,$$

and

$$K(t) = \int_Y \tilde{g}_t(y)\,dy \quad \text{for } t = 0, 1, \ldots, T.$$

Suppose that for some t, equation (VI-20) fails to hold on a set of positive measure

in Y. Dividing $\tilde{g}_t$ by K(t) to obtain a probability density g_t, we observe that $\psi^*(\cdot, t)$ fails to satisfy the necessary conditions of Corollary 2.1.7 of reference [h] for $\psi^*(\cdot, t)$ to maximize probability of detection for cost m(t) for a stationary target with probability density g_t. Thus $\psi^*(\cdot, t)$ is not optimal for cost m(t) for this stationary target problem.

For nonnegative Borel measurable functions f defined on Y, let Q[f] be the probability of detecting a stationary target with location distribution g_t and effort allocation f, i.e.,

$$Q[f] = \int_Y g_t(y)\,(1-e^{-W(y)f(y)})dy.$$

Since $\psi^*(\cdot, t)$ is not optimal for cost m(t), we may find an $f^* \geq 0$ such that $\int_Y f^*(y)dy = m(t)$ and $Q[f^*] > Q[\psi^*(\cdot, t)]$. Observe that

$$\begin{aligned} 1 - P_T[\psi^*] &= \int_Y \tilde{g}_t(y) \exp(-W(y)\psi^*(y,t))dy \\ &= 1 - K(t)\, Q[\psi^*(\cdot, t)] \\ &> 1 - K(t)\, Q[f^*]. \end{aligned}$$

Thus, by taking

$$\psi(y, s) = \begin{cases} \psi^*(y, s) & \text{for } s \neq t \\ f^*(y) & \text{for } s = t, \end{cases}$$

we have $\psi \in \Psi(m)$ and $P_T[\psi] > P_T[\psi^*]$ which contradicts the assumption that ψ^* is T-optimal within $\Psi(m)$. Thus equation (VI-20) must hold and Theorem 2 is proved.

From the definition of $\bar{g}_t(y)$ and $K(t)$, one can see that $g_t = \bar{g}_t/K(t)$ is simply the probability density for the target's location at time t given that it was undetected by the search applied at all times other than t. Let $\gamma(t) = \lambda(t)/K(t)$ for $t = 0, 1, \ldots, T$. Then conditions (VI-20) become

$$g_t(y)\, W(y) e^{-W(y)\psi^*(y,t)} \begin{array}{ll} = \gamma(t) & \text{if } \psi^*(y,t) > 0, \\ \leq \gamma(t) & \text{if } \psi^*(y,t) = 0, \end{array}$$

which are precisely the necessary and sufficient conditions for $\psi^*(\cdot, t)$ to be an optimal allocation of effort for a stationary target with probability density g_t. Thus the optimal moving target plan can be characterized in terms of optimal stationary target plans. That is at each time $t = 0, 1, \ldots, T$, the optimal plan ψ^* allocates the effort available at time t so that $\psi^*(\cdot, t)$ maximizes the probability of detecting a stationary target with probability density g_t.

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